Solutions to Midterm 1

1. (a) (4 points) Let (X, d) be a metric space, and let $A \subset X$. Define what it means for a point $p \in X$ to be in the *interior* of A, and what it means for p to be in the *closure* of A.

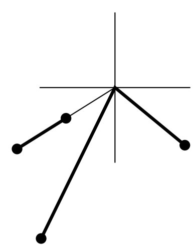
   Solution: To say that $p \in \operatorname{int} A$ means that there is an $r > 0$ such that $B_r(p) \subset A$. To say that $p \in \bar{A}$ means that for every $r > 0$ we have $B_r(p) \cap A = \varnothing$.

   Or if you prefer, $p \in \operatorname{int} A$ means that there is an $r > 0$ such that for all $q \in X$ with $d(p, q) < r$ we have $q \in A$, while $p \in \bar{A}$ means that for all $r > 0$ there is a $q \in X$ with $d(p, q) < r$ and $q \in A$.

   (b) (4 points) Let $X = \mathbb{R}^2$ and let d_* be the SNCF metric, defined by

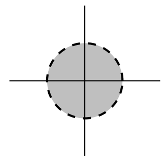
$$d_*(p,q) = \begin{cases} d_2(p,q) & \text{if } p \text{ and } q \text{ lie on the same line} \\ & \text{through the origin, or} \\ d_2(p,0) + d_2(0,q) & \text{otherwise,} \end{cases}$$

   where d_2 is the Euclidean metric. Here's a picture:

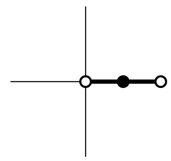


   In this metric, what is the open ball of radius 1 around the origin, and the open ball of radius 1 around the point $(1, 0)$? You don't need to give a proof, but include a picture of each one.

   Solution: The SNCF distance from the origin to any point is the same as the Euclidean distance, so the ball of radius 1 around the origin is the usual one: $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

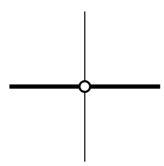


The Euclidean distance from $(1,0)$ to the origin is 1, so any point whose SNCF distance to $(1,0)$ is less than 1 must lie on the same line through the origin, that is, on the x-axis. The ball of radius 1 around $(1,0)$ is $\{(x,y) \in \mathbb{R}^2 : 0 < x < 2 \text{ and } y = 0.\}$.

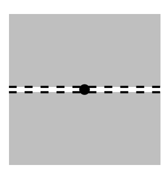


(c) (4 points) Still using the SNCF metric on $\mathbb{R}^2$, let A be the x-axis, and let B be the complement of the x-axis. What is the interior of A? What is the closure of B? Again you don't need to give a proof, but include a picture of each one.

Solution: We know that $\operatorname{int} A \subset A$. From part (b) we see that $(1,0) \in \operatorname{int} A$, and similarly with any point $(x,0)$ as long as $x \neq 0$, but that no ball around the origin is contained in A. Thus $\operatorname{int} A = A \setminus \{(0,0)\}$.



We know that $B \subset \bar{B}$. We see that every ball around the origin meets B, but there is a ball around $(1,0)$ that doesn't meet B, and similarly with every point $(x,0)$ as long as $x \neq 0$. Thus $\bar{B} = B \cup \{(0,0)\}$.



This agrees with the fact that the closure of the complement is the complement of the interior.

(d) (4 points) Let (X, d) be a metric space, and let $A, B \subset X$. Prove that $\operatorname{int}(A \setminus B) = \operatorname{int}(A) \setminus \bar{B}$. You may use anything that was proved in lecture, in the notes, or in the homework.

Solution: We have $A \setminus B = A \cap (X \setminus B)$. In lecture we proved that the interior of the intersection of two sets is the intersection of their interiors, so $\operatorname{int}(A \setminus B) = \operatorname{int}(A) \cap \operatorname{int}(X \setminus B)$. We also proved that the interior of the complement is the complement of the closure, so this becomes $\operatorname{int}(A) \cap (X \setminus \bar{B}) = \operatorname{int}(A) \setminus \bar{B}$.

2. Let (X, d) be a metric space.

(a) (3 points) Define what it means for a sequence $p_1, p_2, p_3, \ldots \in X$ to converge to a limit $\ell \in X$.

Solution: For every $\epsilon > 0$ there is an integer N such that for all $n \geq N$ we have $d(p_n, \ell) < \epsilon$.

(b) (5 points) Suppose we have two convergent sequences $p_n \to \ell$ and $p'_n \to \ell'$. Prove that if $d(p_n, p'_n) < 1$ for all n, then $d(\ell, \ell') \leq 1$.

Solution: First I claim that for all $\epsilon > 0$ we have $d(\ell, \ell') < 1 + \epsilon$. Let $\epsilon > 0$ be given. Because $p_n \to \ell$, there is an N such that for all $n \geq N$ we have $d(p_n, \ell) < \epsilon/2$. Similarly, because $p'_n \to \ell'$ there is an N' such that for all $n \geq N'$ we have $d(p'_n, \ell') < \epsilon/2$. Now if $n \geq \max(N, N')$ then

$$d(\ell, \ell') \leq d(\ell, p_n) + d(p_n, p'_n) + d(p'_n, \ell') < \epsilon/2 + 1 + \epsilon/2,$$

which proves the claim.

Now you can just say this implies that $d(\ell, \ell') \leq 1$, or if you want to give more detail, you can say that if we had $d(\ell, \ell') > 1$, we could take $\epsilon = d(\ell, \ell') - 1$ and get a contradiction.

(c) (3 points) Give an example to show that $d(\ell, \ell') = 1$ can happen in part (b).

Solution: Of course there are many possible answers, but here's one. Let $X = \mathbb{R}$ with the usual metric, let $p_n = 1/n$, and let $p'_n = 1 - 1/n$. Then $d(p_n, p'_n) = |p_n - p'_n| = 1 - 2/n < 1$, but $p_n \to 0$ and $p'_n \to 1$, and $d(0, 1) = 1$.

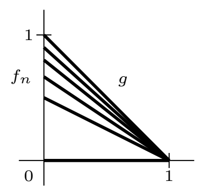
3. (5 points) Let $X = C([0,1])$ with the L^1 metric $d_1(f,g) = \int_0^1 |f-g|$, and let

$$A = \{f \in X : f(x) < 1 \text{ for all } x \in [0,1]\}$$

Prove that A is neither closed nor open. Again you may use anything that was proved in lecture, in the notes, or in the homework.

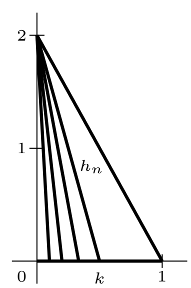
Solution: There are several ways to go about this. I'll prove that A is not closed by producing a sequence in A that converges to a limit that's not in A, and that A is not open by proving that $X \setminus A$ is not closed, producing a sequence in $X \setminus A$ that converges to a limit in A. But you might do it a different way.

Let f_n be the function that goes linearly from $f_n(0) = 1 - 1/n$ to $f_n(1) = 0$, and let $g(x)$ be the function that goes linearly from $g(0) = 1$ to $g(1)$:



We see that $f_n \in A$ but $g \notin A$, and turning our heads we see that $\int_0^1 |f_n - g|$ is the area of a triangle whose base is $1/n$ and whose height is 1, so $d_1(f_n, g) = 1/2n \to 0$, so $f_n \to g$. Thus A is not closed.

Next, let h_n be the function that goes piecewise linearly from $h_n(0) = 2$ to $h_n(1/n) = 0$ to $h_n(1) = 0$, and let k be the function that's constantly zero:



We see that $h_n \in X \setminus A$ but $k \in A$, and that $\int_0^1 |h_n - k|$ is the area of a triangle whose base is $1/n$ and whose height is 2, so $d_1(h_n, k) = 1/n \to 0$, so $h_n \to k$. Thus $X \setminus A$ is not closed, so A is not open.

In lecture next week I'll point out that A is open in the sup metric, and it would have been closed in both the L^1 metric and the sup metric if the definition had said $f(x) \leq 1$ rather than $f(x) < 1$.