Midterm 1

Friday, October 23, 2025

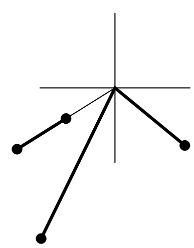
Use your own notebook paper, or get some from me. I encourage you to write in pencil. There are 32 points in total.

1. (a) (4 points) Let (X, d) be a metric space, and let $A \subset X$. Define what it means for a point $p \in X$ to be in the *interior* of A, and what it means for p to be in the *closure* of A.

   (b) (4 points) Let $X = \mathbb{R}^2$ and let d_* be the SNCF metric, defined by

$$d_*(p,q) = \begin{cases} d_2(p,q) & \text{if } p \text{ and } q \text{ lie on the same line through the origin, or} \\ d_2(p,0) + d_2(0,q) & \text{otherwise,} \end{cases}$$

   where d_2 is the Euclidean metric. Here's a picture:



   In this metric, what is the open ball of radius 1 around the origin, and the open ball of radius 1 around the point $(1, 0)$? You don't need to give a proof, but include a picture of each one.

   (c) (4 points) Still using the SNCF metric on $\mathbb{R}^2$, let A be the x-axis, and let B be the complement of the x-axis. What is the interior of A? What is the closure of B? Again you don't need to give a proof, but include a picture of each one.

   (d) (4 points) Let (X, d) be a metric space, and let $A, B \subset X$. Prove that $\mathrm{int}(A \setminus B) = \mathrm{int}(A) \setminus \bar{B}$. You may use anything that was proved in lecture, in the notes, or in the homework.

2. Let (X, d) be a metric space.

   (a) (3 points) Define what it means for a sequence $p_1, p_2, p_3, \ldots \in X$ to converge to a limit $\ell \in X$.

   (b) (5 points) Suppose we have two convergent sequences $p_n \to \ell$ and $p'_n \to \ell'$. Prove that if $d(p_n, p'_n) < 1$ for all n, then $d(\ell, \ell') \leq 1$.

   (c) (3 points) Give an example to show that $d(\ell, \ell') = 1$ can happen in part (b).

3. (5 points) Let $X = C([0,1])$ with the L^1 metric $d_1(f, g) = \int_0^1 |f - g|$, and let

$$A = \{f \in X : f(x) < 1 \text{ for all } x \in [0,1]\}$$

   Prove that A is neither closed nor open. Again you may use anything that was proved in lecture, in the notes, or in the homework.