Let $(X, d)$ be a metric space.

**Definition.** A subset $U \subset X$ is *open* if for every $p \in U$ there is an $\epsilon > 0$ such that the ball $B_\epsilon(p) \subset U$, or equivalently, for all $q \in X$, if $d(p, q) < \epsilon$ then $q \in A$.

**Definition.** A subset $F \subset X$ is *closed* if for every sequence $p_1, p_2, \ldots$ in $F$ that converges to a limit $\ell \in X$, we have $\ell \in F$.

**Proposition.** A subset $A \subset X$ is open if and only if its complement $X \setminus A$ is closed.

We proved this in lecture, but it seemed like we used proof by contradiction more than necessary. Here is a more direct proof:

*Proof.* We will show that $A$ is *not* open if and only if $X \setminus A$ is *not* closed, which is equivalent to the proposition.

First suppose that $A$ is not open. Thus there is a $p \in A$ such that for every $\epsilon > 0$ there is a $q \in X$ with $d(p, q) < \epsilon$ but $q \notin A$. For each positive integer $n$, choose a point $q_n$ such that $d(p, q_n) < 1/n$ but $q_n \notin A$. Then the sequence $q_n$ converges to $p$, and $q_n \in X \setminus A$, but $p \notin X \setminus A$; thus $X \setminus A$ is not closed.

Conversely, suppose that $X \setminus A$ is not closed. Thus there is a convergent quence $p_1, p_2, \ldots \to \ell$ such that $p_n \in X \setminus A$ for all $n$, but $\ell \notin X \setminus A$, that is, $\ell \in A$. For every $\epsilon > 0$ there is an $N$ such that the ball $B_\epsilon(\ell)$ contains $p_n$ for all $n \geq N$, and in particular $B_\epsilon(\ell)$ is not contained in $A$. Thus $A$ is not open. $\square$