Midterm 1

Friday, October 19, 2018

Use your own notebook paper, or get some from me. Put each problem on its own page. Each part is worth 5 points, for a total of 40 points.

1. (a) Let X be a set. Define what it means for a function $d\colon X \times X \to \mathbb{R}$ to be a metric.

   (b) We have often used the square metric on $\mathbb{R}^2$, defined by

   $$d((x_1, x_2),\, (y_1, y_2)) = \max\{|x_1 - y_1|,\, |x_2 - y_2|\}.$$

   Why is

   $$d((x_1, x_2),\, (y_1, y_2)) = \min\{|x_1 - y_1|,\, |x_2 - y_2|\}$$

   not a metric? (Give an example to show that one of the axioms you listed in part (a) fails.)

2. (a) Let (X, d) be a metric space and $x_1, x_2, \ldots$ a sequence in X. Define what it means for the sequence to converge to a limit $\ell \in X$.

   (b) Suppose that d is a discrete metric:

   $$d(p, q) = \begin{cases} 0 & \text{if } p = q, \\ 1 & \text{if } p \neq q. \end{cases}$$

   Show that every convergent sequence is eventually constant: that is, there is an integer N such that $x_n = \ell$ for all $n \geq N$.

3. (a) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a function $f\colon X \to Y$ to be continuous at a point $p \in X$. (Use δ and ϵ, not open sets.)

   (b) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1$ is continuous at every point. Use the usual metric on $\mathbb{R}$, and either the taxicab, Euclidean, or square metric on $\mathbb{R}^2$ (your choice). Hint: take $\delta = \epsilon$.

4. (a) Let (X, d) be a metric space. Define what it means for a subset $U \subset X$ to be open.

   (b) Let $p \in X$ and $r > 0$. Show that the ball

$$B_r(p) = \{q \in X : d(p, q) < r\}$$

   is an open set. Hint: Use the triangle inequality.