

Here is a streamlined account of the proof we came up with in class Monday. If I had it to do over again I'd pick slightly different notation.

Definition. A topological space X is *disconnected* if we can write $X = U \cup V$, where U and V are open and non-empty, and $U \cap V = \emptyset$. The space X is *connected* if it is not disconnected.

Definition. A subset $A \subset \mathbb{R}$ is an *interval* if for all $a, b, c \in \mathbb{R}$ with $a < b < c$, if $a \in A$ and $c \in A$ then $b \in A$.

Theorem. A subset $A \subset \mathbb{R}$ is connected if and only if it is an interval.

Proof. First let us show that if A is not an interval then A is disconnected. Let $a < b < c$ with $a, c \in A$ and $b \notin A$. Let $U = (-\infty, b) \cap A$ and $V = (b, \infty) \cap A$. Then U and V are non-empty, disjoint, open in A , and $A = U \cup V$.

Conversely, let us show that if A is disconnected then A is not an interval. Write $A = U \cup V$, where U and V are non-empty, disjoint, and open in A . Observe that U and V are also closed in A , because their complements $A \setminus U = V$ and $A \setminus V = U$ are open in A .

Let $a \in U$ and $c \in V$. Assume that $a < c$; if $a > c$ then the argument is similar. We want to produce some $b \notin A$ with $a < b < c$.

Let

$$b = \sup\{u \in U : u < c\}.$$

We have $a \leq b \leq c$. If $b \notin A$ then we are done. So assume that $b \in A$; then $b \in U$, because b is in the closure of U and U is closed in A .

Let

$$b' = \inf\{v \in V : b < v\}.$$

We have $a \leq b \leq b' \leq c$. If $b' \notin A$ then we are done. So assume that $b' \in A$; then $b' \in V$, because b' is in the closure of V and V is closed in A .

Because $b \in U$ and $b' \in V$, we have $b \neq b'$, so $b < b'$, so we can consider

$$b'' = \frac{b + b'}{2},$$

which satisfies

$$a \leq b < b'' < b' \leq c.$$

To finish the proof, we show that $b'' \notin A$: if b'' were in U then we would have $b'' \leq b$, by construction of b as the sup of $\{u \in U : u < c\}$; and if b'' were in V then we would have $b' \leq b''$, by construction of b' as the inf of $\{v \in V : b < v\}$. $\square$