Homework 9

Due Monday, November 26, 2018

1. Let X be a topological space. Define a relation $\sim$ on X by declaring that $p \sim q$ iff there is a conected subspace $A \subset X$ containing p and q.

   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

   (b) The equivalence classes of this equivalence relation are called are called *connected components*. Describe (without proof) the connected components of the following spaces:

      i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.

      ii. $\mathbb{Q}$.

      iii. The topologist's sine curve.

   (c) Let $p \in X$, let P be the path component of p, and let C be the connected component of p. Show that $P \subset C$.

2. (a) Let X be a connected space, and let $\sim$ be an equivalence relation on X. Show that if the equivalence classes of $\sim$ are open then every point in X is equivalent to every other point.

   (b) Show that a connected open subset $U \subset \mathbb{R}^n$ is path-connected. Hint: Show that the path components of U are open.

3. Which of the following topologies on $\mathbb{R}$ are compact? Give a proof either way.

   (a) The finite complement topology.

   (b) The topology where U is open iff either $U = \varnothing$ or $0 \in U$.

   (c) The lower semi-continuous topology.

4. A continuous map $f\colon X \to Y$ is called *proper* if the preimage of any compact set $K \subset Y$ is compact.

   (a) Show that the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is proper.

   (b) Show that the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$ is not proper.

   (c) Show that if f is proper then the preimage of every point is compact.

   (d) Give an example of a continuous map $f\colon X \to Y$ for which the preimage of every point is compact, but nonetheless f is not proper.

   (e) Show that if X is compact and Y is Hausdorff then any continuous map $f\colon X \to Y$ is proper.

   (f) Let X and Y be topological spaces. Show that the projection $p\colon X \times Y \to X$ is proper if and only if Y is compact.

5. What is one question you have about last week's lectures?