

Homework 7

Due Friday, November 9, 2018

1. In lecture we saw that a space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed. Describe the closure of the diagonal in $\mathbb{R}^2$ for *one* of the following non-Hausdorff topologies on $\mathbb{R}$:
 - (a) From homework 3 #4, the finite complement topology: $U \subset \mathbb{R}$ is open if either $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or F is finite.
 - (b) From homework 3 #5: $U \subset \mathbb{R}$ is open if either $U = \emptyset$ or $0 \in U$; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or $0 \notin F$.
 - (c) From homework 5 #1, the lower semi-continuous topology: the open sets are $\emptyset$, all of $\mathbb{R}$, and sets of the form (a, ∞) for $a \in \mathbb{R}$; equivalently, the closed sets are $\emptyset$, all of $\mathbb{R}$, and sets of the form $(-\infty, a]$ for $a \in \mathbb{R}$.
2. Let X and Y be topological spaces, and let $f, g: X \rightarrow Y$ be two continuous maps. Show that if Y is Hausdorff then the set

$$E = \{x \in X : f(x) = g(x)\}$$

is closed. (Hint: Think about $f \times g: X \rightarrow Y \times Y$.)

3. Let X be a topological space. A subset $A \subset X$ is called *dense* if $\bar{A} = X$.
 - (a) Which of the following is dense in $\mathbb{R}$ with the usual topology? With the finite complement topology? (No proofs.)
 - (i) $\mathbb{Q}$. (ii) $\mathbb{R} \setminus \mathbb{Q}$. (iii) $\mathbb{Z}$. (iv) $\mathbb{R} \setminus \mathbb{Z}$.
 - (b) Show that $A \subset X$ is dense if and only if every non-empty open $U \subset X$ meets A , that is, $A \cap U \neq \emptyset$.

- (c) Suppose that $A \subset X$ is dense, and Y is Hausdorff. Show that two continuous maps $f, g: X \rightarrow Y$ that agree on A must agree on all of X : that is, if $f(a) = g(a)$ for all $a \in A$ then $f(x) = g(x)$ for all $x \in X$. Hint: Use #2.
- (d) Give a counterexample to (c) when Y is not Hausdorff.
(One possibility is let $X = \mathbb{R}$ with the usual topology, and $Y = \mathbb{R}$ with one of the topologies from problem 1.)
4. Show that $X = [0, 1)$ is not homeomorphic to $Y = (0, 1)$, with the usual topologies on both.
Hint: A homeomorphism $f: X \rightarrow Y$ would induce a homeomorphism from $X \setminus \{0\}$ to $Y \setminus \{f(0)\}$.
5. Suppose that $A \subset X$ is connected. Show that $\bar{A}$ is connected.
Hint: Suppose we could write $\bar{A} = F \cup G$, where F and G are non-empty, *closed* in $\bar{A}$, and $F \cap G = \emptyset$.
6. Let X be a set.
- An *equivalence relation* on X is a relation $\sim$ that is reflexive ($x \sim x$), symmetric (if $x \sim y$ then $y \sim x$), and transitive (if $x \sim y$ and $y \sim z$ then $x \sim z$).
 - A *partition* of X is a collection of disjoint subsets of X whose union is all of X .
 - A *surjection* from X to another set is an “onto” map.
- (a) Show that an equivalence relation on X determines a partition of X , and vice versa.
- (b) Show that an equivalence relation on X determines a surjection onto another set, and vice versa.
7. What is one question you have about last week’s lectures?