Final Exam

Math 420

June 10, 2016

Use your own notebook paper. Start each problem on a new sheet of paper.

1. Consider the linear system $X' = AX$, where

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

   (a) Find the eigenvalues of A.

   (b) Find the eigenvectors of A.

   (c) In $\mathbb{R}^3$, sketch the "stable subspace" spanned by the eigenvector(s) with negative eigenvalue(s). Sketch the flow of the linear system in this subspace *only*.

   (d) On the same sketch, show the "unstable subspace" spanned by the eigenvector(s) with *positive* eigenvalue(s), and sketch the flow of the linear system in *this* subspace.

   (e) Complete the sketch by adding a few flow lines outside of those two subspaces.

2. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   (a) Find $\exp(A)$ and $\exp(B)$.
   Hint: Just do it by hand, using the power series.

   (b) Find $\exp(A + B)$.

   (c) Find $\exp(A)\exp(B)$ and $\exp(B)\exp(A)$. Observe that these are different from one another, and from $\exp(A + B)$.

   (d) State a sufficient condition for square matrices M and N to satisfy $\exp(M)\exp(N) = \exp(M + N) = \exp(N)\exp(M)$.

3. From reading and lecture: Consider the non-linear system

$$\begin{aligned} x' &= \frac{1}{2}x - y - \frac{1}{2}x(x^2 + y^2) \\ y' &= x + \frac{1}{2}y - \frac{1}{2}y(x^2 + y^2). \end{aligned}$$

   (a) Substitute $x = r\cos\theta$, $y = r\sin\theta$ into these equations.

   (b) Take $\cos\theta$ times the first equation from (a), plus $\sin\theta$ times the second, and simplify to get a differential equation in r alone.

   (c) Take $\cos\theta$ times the *second* equation from (a), *minus* $\sin\theta$ times the *first*, and simplify to get a very simple differential equation in θ alone.

$$r\theta' = r.$$

   (d) Look at part (b). For which values of r is r increasing? Decreasing? Constant? Looking at part (c), what can you say about θ?

   (e) Sketch the flow of the system.
   Hint: Something special happens at $r = 1$.

4. Consider the non-linear system

$$\begin{aligned} x' &= x - y^2 \\ y' &= y - x^2. \end{aligned}$$

   (a) There is an equilibrium point at the origin. What is the linearization there? Sketch the flow of the linearization.

   (b) There is one more equilibrium point. What is it?

   (c) What is the linearization at this second equilibrium point? (That is, take the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}$$

   and plug in the point.) What are the eigenvalues and eigenvectors? Sketch the flow of the linearization.

   (d) Notice that the system is symmetric under reflecting across the diagonal line $y = x$. What does the flow of the non-linear system look like along that line?

   (e) Just for fun, see which way the vector field is pointing at the four points $(\pm 10, \pm 10)$.

   (f) Sketch, very roughly, what the flow might look like.