Worksheet 22

Math 392, Abstract Algebra

Wednesday, March 10, 2021

On Monday we were working through a proof of §6.3 Theorem 3.7, which many of you had questions about. We argued that if G is a group of order 6, then it must have an element a of order 3 and an element b of order 2.

Saying that $|a| = 3$ means that 1, a, and a^2 are all distinct, and $a^3 = 1$.

Convince yourselves that the following statements are true:

1. b is not equal to 1, a, or a^2.
2. ba is not equal to 1, a, a^2, or b.
3. ba^2 is not equal to 1, a, a^2, b, or ba.

   Thus we've found all six elements of G:

   $$G = \{1, a, a^2, b, ba, ba^2\}.$$

4. ab is not equal to 1, a, a^2, or b. (This is entirely similar to #2.)

   Thus either $ab = ba$ or $ab = ba^2$.

5. If $ab = ba$, then the map $\phi\colon \mathbb{Z}_3 \times \mathbb{Z}_2 \to G$ given by $\phi(\bar{m}, \bar{n}) = a^m b^n$ is an isomorphism.
6. If $ab = ba^2$, then G is isomorphic to the dihedral group D_3, just by renaming a to r and b to s:

   $$D_3 = \{1, r, r^2, s, sr, sr^2\} \qquad \text{where } rs = sr^2.$$

7. If you have time, convince yourselves that $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ and $D_3 \cong S_3$, so any group of order 6 is isomorphic to either $\mathbb{Z}_6$ or S_3.