Worksheet 18

Math 392, Abstract Algebra

Friday, February 26, 2021

This is about the symmetric group S_4, also called the group of permutations of the set $\{1, 2, 3, 4\}$.

1. Let σ be the "cycle" that send everything around in a loop from 1 to 2 to 3 to 4 and back to 1:

$$\sigma\colon \qquad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[lll, bend left] \end{tikzcd}$$

   Do you agree that σ has order 4?

2. For $i = 1$, 2, or 3, let τ_i be the "transposition" that switches i with $i+1$ and leaves the rest alone:

$$\tau_1\colon \qquad \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] & 3 \arrow[loop right] & 4 \arrow[loop right] \end{tikzcd}$$

$$\tau_2\colon \qquad \begin{tikzcd} 1 \arrow[loop right] & 2 \arrow[r, bend left] & 3 \arrow[l, bend left] & 4 \arrow[loop right] \end{tikzcd}$$

$$\tau_3\colon \qquad \begin{tikzcd} 1 \arrow[loop right] & 2 \arrow[loop right] & 3 \arrow[r, bend left] & 4 \arrow[l, bend left] \end{tikzcd}$$

   You can write σ as a product of τ_1, τ_2, and τ_3 in some order. What is it?

3. Let σ' be the cycle that sends everything around in a loop from 1 to 3 to 2 to 4 and back to 1:

$$\sigma'\colon \qquad \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[rr, bend left] & 3 \arrow[l] & 4 \arrow[lll, bend left] \end{tikzcd}$$

   Write σ' as a product of τ_1, τ_2, and τ_3 in a different order.

4. Check that $\tau_1 \circ \tau_2 \circ \tau_1 = \tau_2 \circ \tau_1 \circ \tau_2$, which is sometimes called the "braid relation." Check that $\tau_1 \circ \tau_3 = \tau_3 \circ \tau_1$.

5. Challenge: If you take the description of σ above and switch 2 with 3, then you get σ'. Check that $\tau_2 \circ \sigma \circ \tau_2^{-1} = \sigma'$.

   Also check that $\sigma \circ \tau_1 \circ \sigma^{-1} = \tau_2$. In what sense does σ take the description of τ_1 and turn it into τ_2?