Worksheet 10

Math 392, Abstract Algebra

Wednesday, February 3, 2021

Today in lecture we tried to understand $\mathbb{Z}[i]/\langle p\rangle$ by writing

$$\mathbb{Z}[i]/\langle p\rangle \;\cong\; \mathbb{Z}[x]/\langle x^2+1, p\rangle \;\cong\; \mathbb{Z}_p[x]/\langle x^2+1\rangle.$$

1. In the past we've proved that

   $$\mathbb{Z}[\sqrt{-5}]/\langle 2,\, 1+\sqrt{-5}\rangle \;\cong\; \mathbb{Z}_2$$

   by writing down a surjective homomorphism $\phi\colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}_2$ with $\ker\phi = \langle 2,\, 1+\sqrt{-5}\rangle$ and quoting the first isomorphism theorem.

   But today's technique gives another way to understand it. Discuss each of the following steps:

   $$\begin{aligned}\mathbb{Z}[\sqrt{-5}]/\langle 2,\, 1+\sqrt{-5}\rangle &\cong \mathbb{Z}[x]/\langle x^2+5\,,2,\, 1+x\rangle\\ &\cong \mathbb{Z}_2[x]/\langle x^2+\bar{5},\, \bar{1}+x\rangle\\ &\cong \mathbb{Z}_2[x]/\langle x+\bar{1}\rangle\\ &\cong \mathbb{Z}_2\end{aligned}$$

   (Going from the second line to the third, notice that in $\mathbb{Z}_2[x]$ we have

   $$x^2+\bar{5} = x^2+\bar{1} = (x+\bar{1})^2,$$

   so the ideals $\langle x^2+\bar{5},\, \bar{1}+x\rangle$ and $\langle x+\bar{1}\rangle$ are the same.)

2. Play the same game with

   $$\mathbb{Z}[\sqrt{-5}]/\langle 3,\, 1+\sqrt{-5}\rangle \;\cong\; \mathbb{Z}_3$$

   and

   $$\mathbb{Z}[\sqrt{-5}]/\langle 3,\, 1-\sqrt{-5}\rangle \;\cong\; \mathbb{Z}_3$$

3. Challenge: Play the same game with

   $$\mathbb{Z}[\sqrt{-5}]/\langle 1+\sqrt{-5}\rangle \;\cong\; \mathbb{Z}_6.$$