Worksheet 1

Math 392, Abstract Algebra

Monday, January 4, 2021

0. Say hello to your group. You will be working with them all week. Did they have a good break?

1. Let R be a commutative ring, and let $I, J \subset R$ be ideals.

   (Recall that this means that for all $a, b \in I$ and all $r \in R$ we have $a + b \in I$ and $ra \in I$, and similarly with J.)

   Prove that the intersection $I \cap J$ is an ideal.

2. In $\mathbb{Z}$, what is $\langle 6 \rangle \cap \langle 10 \rangle$?

   (Recall that $\langle 6 \rangle$ is the set of all multiples of 6, and similarly with $\langle 10 \rangle$.)

3. Given two ideals $I, J \subset R$, define

   $$I + J = \{a + b : a \in I, b \in J\}.$$

   Show that $I + J$ is an ideal.

4. In $\mathbb{Z}$, what is $\langle 6 \rangle + \langle 10 \rangle$?

5. Challenge: Given two ideals $I, J \subset R$, define

   $$I \cdot J = \{a_1 b_1 + \cdots + a_k b_k : a_1, \ldots, a_k \in I,\ b_1, \ldots, b_k \in J\}.$$

   (Here the number of summands k is allowed to vary, so if you want two arbitrary elements of $I \cdot J$ then you should call them $a_1 b_1 + \cdots + a_k b_k$ and $c_1 d_1 + \cdots + c_l d_l$.)

   Show that $I \cdot J$ is an ideal.

   In $\mathbb{Z}$, what is $\langle 6 \rangle \cdot \langle 10 \rangle$?

   (If you don't get to this problem, we'll take it up on Wednesday.)