Solutions to Homework 7

Continued from Worksheet 12, which accompanied Lecture 16. Let G be the group of rotations of the cube. We have seen that $|G| = 24$.

1. How many elements of order 4 are there in G? That is, how many 90° rotations? Describe them.

   Solution: 6. For each of the three pairs of opposite faces, take the line that connects their centers, and rotate around that line 90° in either direction.

2. How many elements of order 3, or 120° rotations? Describe them.

   Solution: 8. For each of the four pairs of opposite vertices, take the line that connects them, and rotate around that line 120° in either direction.

3. How many elements of order 2, or 180° rotations? Describe them. Notice that there are two different kinds.

   Solution: 9. First, for each of the three pairs of opposite faces, take the line that connects their centers, and rotate around that line 180°. Second, for each of the six pairs of opposite edges, take the line that connects their midpoints, and rotate around that line 180°.

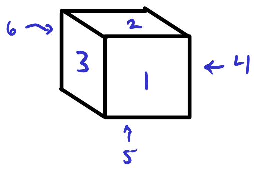
4. How many elements of order 1? Do your numbers add up to 24?

   Solution: In any group there is one element of order 1: the identity. We have $6 + 8 + 9 + 1 = 24$.

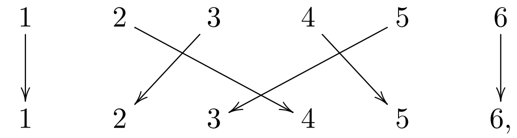
5. Label the six faces of a cube as 1, 2, 3, 4, 5, 6 in whatever way you want. Choose some 90° rotation. It gives a permutation of the six faces: maybe it moves face 1 to where face 2 used to be, face 2 to where face 4 used to be, and so on. Write down the permutation, using whatever notation you like. Of course your answer will depend on how you labeled the faces, and on which rotation you chose.

   (Sanity check: Does your permutation have order 4?)

   Solution: I'll number the faces as on standard right-handed dice:



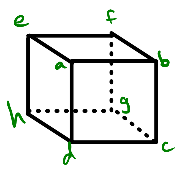
   If I rotate face 1 clockwise, then the permutation is



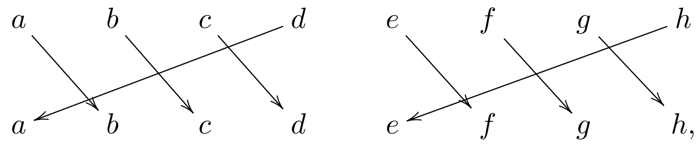
   or in cycle notation, $(2\,4\,5\,3)$.

6. Label the eight vertices as a, b, c, d, e, f, g, h, again in whatever way you want. The same 90° rotation permutes these as well. Write down the permutation.

   Solution: I'll label the vertices as shown:



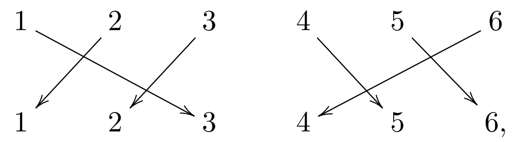
   Then the permutation is



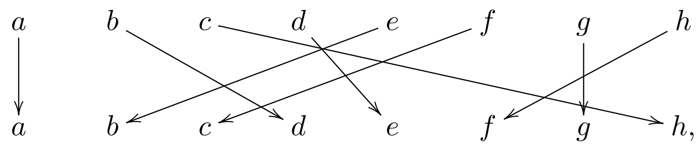
   or in cycle notation, $(a\,b\,c\,d)(e\,f\,g\,h)$.

7. Same with a 120° rotation: choose one, and say how it permutes the faces and the vertices. Do your permutations have order 3?

   Solution: I'll take the vertex shared by faces 1, 2, and 3, and rotate 120° clockwise. The permutation of the faces is



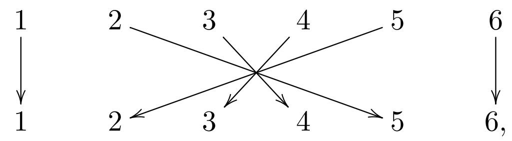
   or (1 3 2)(4 5 6), and the permutation of the vertices is



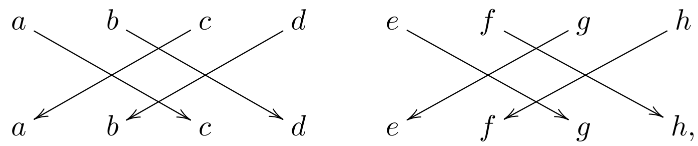
   or in cycle notation, $(b\,d\,e)(c\,h\,f)$.

8. Same with a 180° rotation – actually, do one of each kind. Do your permutations have order 2?

   Solution: If we take the 180° rotation around face 1, then the permutation of the faces is

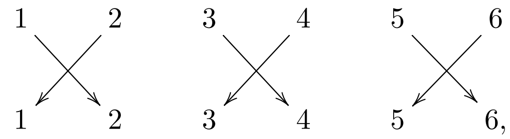


   or in cycle notation, (2 5)(4 3); and the permutation of the vertices is

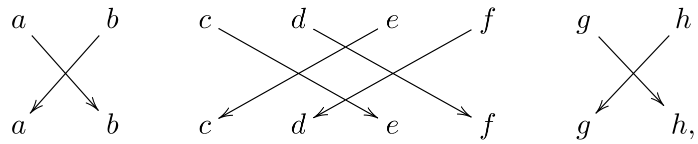


   or in cycle notation, $(a\,c)(b\,d)(e\,g)(f\,h)$.

If we take edge shared by faces 1 and 2, the opposite edge shared by faces 5 and 6, and rotate 180° around the line joining their centers, then the permutation of the faces is



or in cycle notation, $(1\,2)(3\,4)(5\,6)$, and the permutation of the vertices i



or in cycle notation, $(a\,b)(c\,d)(d\,f)(g\,h)$.

§6.2 #12. *Let $a \in g$ be fixed, and define $\phi\colon G \to G$ by $\phi(x) = axa^{-1}$ for $x \in G$. Prove that ϕ is a homomorphism. Under what circumstances is ϕ an isomorphism?*

Solution: For $x, y \in G$ we have

$$\phi(x)\phi(y) = axa^{-1} \cdot aya^{-1} = axya^{-1} = \phi(xy),$$

so ϕ is a homomorphism.

It is always an isomorphism, with inverse is given by

$$\psi(x) = a^{-1}xa.$$

We have

$$\psi(\phi(x)) = a^{-1}(axa^{-1})a = x$$

and similarly $\phi(\psi(x)) = x$, so ϕ and ψ are inverse bijections.