# Homework 3

Due Friday, January 22, 2021

**1.** Let $R = \mathbb{Q}[x]$.

a. Let $I = \langle x^2 \rangle$. For $f, g \in R$, prove that $f \equiv g \pmod{I}$ if and only if $f(0) = g(0)$ and $f'(0) = g'(0)$.

b. Let $J = \langle (x-5)^2 \rangle$. For $f, g \in R$, prove that $f \equiv g \pmod{J}$ if and only if $f(5) = g(5)$ and $f'(5) = g'(5)$.

c. Prove that the map $\phi\colon R \to \mathbb{Q} \times \mathbb{Q}$ given by $\phi(f) = (f(5), f(6))$ is a surjective homomorphism, and that $\ker \phi = \langle x^2 - 11x + 30 \rangle$.

(The Cartesian product of two rings appeared on Worksheet 22 [which accompanied Lecture 25] last quarter, and on page 127 of the text.)

(Observe that $x^2 - 11x + 30 = (x-5)(x-6)$.)

d. Prove that the map $\psi\colon R \to \mathbb{Q} \times \mathbb{Q}$ given by $\psi(f) = (f(5), f'(5))$ is not a homomorphism.

**2.** Continued from Worksheet 6: Let $R = \mathbb{Z}[x]$, and let $I = \langle 2, x^2 + 5 \rangle$.

a. Prove that $(x+1)(x-1) \in I$.

b. Prove that $x + 1 \notin I$, as follows. If $x + 1$ were in $I$ then we could write

$$x + 1 = 2f + (x^2 + 5)g$$

for some $f, g \in R$. Consider the reduction homomorphism $\rho\colon R \to \mathbb{Z}_2[x]$ from example 1(d) on page 115, which takes a polynomial $a_n x^n + \cdots + a_0 \in R$ to $\bar{a}_n x^n + \cdots + \bar{a}_0 \in \mathbb{Z}_2[x]$. Apply $\rho$ to the displayed equation above to get a contradiction.

(Notice that $\rho(2) = \bar{0}$, so $\rho(2f) = \bar{0}$.)

c. Write "Similarly, $x - 1 \notin I$. Thus $I$ is not a prime ideal."

**3.** Based on §4.2 #4:

a. Prove that if $\phi\colon R \to S$ and $\psi\colon S \to T$ are ring homomorphisms, then so is their composition $\psi \circ \phi\colon R \to T$.

b. Prove that if $\phi\colon R \to S$ is a ring isomorphism, then so is its inverse $\phi^{-1}\colon S \to R$.