# Solutions to Homework 2

**1.** Let $R$ be a commutative ring, and let $I, J \subset R$ be ideals. We have seen that $I \cap J$ is also an ideal.

a. Give an example to show that $I \cup J$ need not be an ideal.

   **Solution:** Let $R = \mathbb{Z}$, let $I = \langle 6 \rangle$, and let $J = \langle 10 \rangle$. Then $I \cup J$ is not closed under addition: we have $6 \in I \cup J$, and $10 \in I \cup J$, but $6 + 10 = 16$ is neither in $I$ nor in $J$, so it is not in $I \cup J$.

b. Show that if $K$ is an ideal with $I \subset K$ and $J \subset K$, then $I + J \subset K$.

   (So $I + J$ is the smallest ideal that contains $I \cup J$.)

   **Solution:** A general element of $I + J$ can be written as $a + b$, where $a \in I$ and $b \in J$. Because $I \subset K$, we have $a \in K$, and because $J \subset K$ we have $b \in K$. Because $K$ is an ideal, it is closed under addition, so $a + b \in K$. Thus $I + J \subset K$ as desired.

c. On homework 1 you showed that having one ideal contained in another generalizes having one number be divisible by the other (in a somewhat confusingly backwards way). In lecture we've seen that the sum of two ideals $I + J$ generalizes the gcd of two numbers. What down-to-earth statement about gcds and divisibility does the statement in part (b) generalize?

   **Solution:** If $a \mid b$ and $a \mid c$ then $a \mid \gcd(b, c)$.

**2.** Let $R = \mathbb{Z}[\sqrt{-5}]$. In the last week of math 391 we saw that the map $\phi\colon R \to \mathbb{Z}_2$ defined by $\phi(a + b\sqrt{-5}) = \bar{a} + \bar{b}$ was a ring homomorphism, and that $\ker \phi = \langle 2, 1 + \sqrt{-5}\rangle$.

a. Show that the map $\psi\colon R \to \mathbb{Z}_3$ given by

$$\psi(a + b\sqrt{-5}) = \bar{a} - \bar{b}$$

is a ring homomorphism.

**Solution:** First, we have $\psi(1) = \psi(1 + 0\sqrt{-5}) = \bar{1} - \bar{0} = \bar{1}$.

Next, let $z = a + b\sqrt{-5}$ and $w = c + d\sqrt{-5}$. Then we see that

$$z + w = (a + c) + (b + d)\sqrt{-5},$$

so

$$\begin{aligned}\psi(z + w) &= \overline{a + b} - \overline{c + d}\\ &= \bar{a} + \bar{c} - \bar{b} - \bar{d}\\ &= \bar{a} - \bar{b} + \bar{c} - \bar{d}\\ &= \psi(z) + \psi(w).\end{aligned}$$

Finally, we find that

$$zw = (ab - 5cd) + (ac + bd)\sqrt{-5},$$

$$\begin{aligned}\psi(zw) &= \overline{ab - 5cd} - \overline{ac + bd}\\ &= \overline{ad} + \overline{cd} - \overline{ac} - \overline{bd}\\ &= (\bar{a} - \bar{b})(\bar{c} - \bar{d})\\ &= \psi(z)\psi(w),\end{aligned}$$

where in the second line we used the fact that $-\bar{5} = \bar{1}$ in $\mathbb{Z}_3$.

b. Show that $\ker \psi = \langle 3, 1 + \sqrt{-5}\rangle$.

Hint: One containment is easy if you use problem 1 from homework 1.

**Solution:** We have $\psi(3) = \bar{3} - \bar{0} = \bar{0}$ and $\psi(1 + \sqrt{-5}) = \bar{1} - \bar{1} = 0$, so $3 \in \ker \psi$ and $1 + \sqrt{-5} \in \ker \psi$, so $\langle 3, 1 + \sqrt{-5}\rangle \subset \ker \psi$ by problem 1 from homework 1.

For the reverse containment $\ker\psi \subset \langle 3, 1+\sqrt{-5}\rangle$, suppose that $z = a + b\sqrt{-5}$ is in $\ker\psi$. Then $\psi(z) = \bar{a} - \bar{b} = \bar{0}$ in $\mathbb{Z}_3$, so $a \equiv b \pmod 3$. Write $b = a + 3c$ for some $c \in \mathbb{Z}$; then we have

$$\begin{aligned} z &= a + (a+3c)\sqrt{-5} \\ &= a(1+\sqrt{-5}) + 3c\sqrt{-5}, \end{aligned}$$

so we see that $z \in \langle 3, 1+\sqrt{-5}\rangle$.

c. In lecture last week I claimed that 1 and $-1$ are not in $\langle 3, 1+\sqrt{-5}\rangle$. Prove this by showing that they're not in $\ker\psi$.

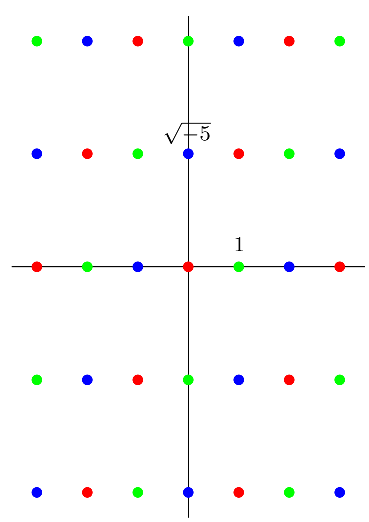
Hint: Don't work hard.

**Solution:** We have $\psi(1) = \bar{1} \neq \bar{0}$, so $1 \notin \ker\psi$. Similarly, $\psi(-1) = \overline{-1} = \bar{2} \neq \bar{0}$, so $-1 \notin \ker\psi$.

d. Draw a picture of $R$ as a lattice of dots in $\mathbb{C}$, going at least 3 dots in every direction. For each point $z$ in your lattice, color it red if $\psi(z) = \bar{0}$, green if $\psi(z) = \bar{1}$, and blue if $\psi(z) = \bar{2}$; or choose different colors according to your own taste.

(What is this about? Two points $z, w \in R$ are the same color if and only if $z \equiv w \pmod I$, where $I = \ker\psi = \langle 3, 1+\sqrt{-5}\rangle$. So you're color-coding the equivalence classes mod $I$. In particular, the red points are $I$ itself.)

**Solution:** (A hand-drawn picture is perfectly acceptable here, even if you typed the rest of the assignment.)



**§4.1 #8.** *Prove that a ring homomorphism $\phi\colon R \to S$ is injective if and only if* $\ker \phi = \langle 0 \rangle$.

Observe that the ideal $\langle 0 \rangle$ is just the one-element subset $\{0\} \subset R$.

First suppose that $\phi$ is injective. Let $r \in \ker \phi$; then $\phi(r) = 0$, but also $\phi(0) = 0$ as we proved last term, so $r = 0$ because $\phi$ is injective. Thus $\ker \phi = \langle 0 \rangle$.

Conversely, suppose that $\ker \phi = \langle 0 \rangle$. If two elements $x, y \in R$ satisfy $\phi(x) = \phi(y)$, then $\phi(x - y) = \phi(x) - \phi(y) = 0$, so $x - y \in \ker \phi$, so $x - y = 0$, so $x = y$. Thus $\phi$ is injective.

**§4.1 #16.** *Let $R$ and $S$ Be commutative rings, and let $\phi\colon R \to S$ be a ring homomorphism.*

a. *Given an ideal $J \subset S$, define*

$$\mathcal{J} = \{a \in R : \phi(a) \in J\} \subset R.$$

*Prove that $\mathcal{J}$ is an ideal.*

Let $a, b \in \mathcal{J}$ and $r \in R$ be given. We have $\phi(a) \in J$ and $\phi(b) \in J$ by the definition of $\mathcal{J}$, so $\phi(a + b) = \phi(a) + \phi(b)$ is in $J$ because $J$ is closed under addition, so $a + b \in \mathcal{J}$. We have have $\phi(r) \in S$ and still $\phi(a) \in J$, so $\phi(ra) = \phi(r)\phi(a)$ is in $J$ because $J$ is an ideal, so $ra \in \mathcal{J}$.

[The grader points out that I should also have shown that $\mathcal{J}$ is non-empty. We have $0_S \in J$, and $\phi(0_R) = 0_S$, so $0_R \in \mathcal{J}$.]

b. *Given an ideal $I \subset R$, define*

$$\mathcal{I} = \{\phi(a) : a \in I\} \subset S.$$

*Prove that if $\phi$ is surjective then $\mathcal{I}$ is an ideal. Give an example to show that if $\phi$ is not surjective then $\mathcal{I}$ need not be an ideal.*

Given two elements of $\mathcal{I}$, write them as $\phi(a)$ and $\phi(b)$ for some $a, b \in I$. Then $a + b \in I$ because $I$ is an ideal, so $\phi(a) + \phi(b) = \phi(a + b)$ is in $\mathcal{I}$. Let $s \in S$; if $\phi$ is surjective, then we can write $s = \phi(r)$ for some $r \in R$. Then $ra \in I$ because $I$ is an ideal, so $s \cdot \phi(a) = \phi(r)\phi(a) = \phi(ra)$ is in $\mathcal{I}$.

[Again, I should have shown that $\mathcal{I}$ is non-empty. We have $0_R \in I$, and $\phi(0_R) = O_S$, so $0_S \in \mathcal{I}$.]

For a counterexample when $\phi$ is not surjective, let $R = \mathbb{Z}$, let $S = \mathbb{Q}$, let $\phi$ be the inclusion map, and let $I = \langle 2 \rangle \subset \mathbb{Z}$, the set of all even numbers. Then $\mathcal{I} \subset \mathbb{Q}$ is still the set of all even integers, but it is not an ideal in $\mathbb{Q}$, because $\frac{1}{2} \in \mathbb{Q}$ and $2 \in \mathcal{I}$ but $\frac{1}{2} \cdot 2 = 1 \notin \mathcal{I}$.

c. *As a particular case of b., prove that if $\phi$ maps onto $S$ and $I = \langle a \rangle \subset R$, then $\mathcal{I} = \langle \phi(a) \rangle \subset S$.*

We have $a \in I$, so $\phi(a) \in \mathcal{I}$, so $\langle \phi(a) \rangle \subset \mathcal{I}$ by problem 1 on homework 1. For the reverse inclusion, every element of $I$ can be written as $ra$ for some $r \in R$, so every element of $\mathcal{I}$ can be written as $\phi(ra)$, but this equals $\phi(r)\phi(a)$, so it is an element of $\langle \phi(a) \rangle$.