Homework 2

Due Friday, January 15, 2021

1. Let R be a commutative ring, and let $I, J \subset R$ be ideals. We have seen that $I \cap J$ is also an ideal.

a. Give an example to show that $I \cup J$ need not be an ideal.

b. Show that if K is an ideal with $I \subset K$ and $J \subset K$, then $I + J \subset K$.

   (So $I + J$ is the smallest ideal that contains $I \cup J$.)

c. On homework 1 you showed that having one ideal contained in another generalizes having one number be divisible by the other (in a somewhat confusingly backwards way). In lecture we've seen that the sum of two ideals $I + J$ generalizes the gcd of two numbers. What down-to-earth statement about gcds and divisibility does the statement in part (b) generalize?

2. Let $R = \mathbb{Z}[\sqrt{-5}]$. In the last week of math 391 we saw that the map $\phi \colon R \to \mathbb{Z}_2$ defined by $\phi(a + b\sqrt{-5}) = \bar{a} + \bar{b}$ was a ring homomorphism, and that $\ker \phi = \langle 2, 1 + \sqrt{-5} \rangle$.

a. Show that the map $\psi \colon R \to \mathbb{Z}_3$ given by

$$\psi(a + b\sqrt{-5}) = \bar{a} - \bar{b}$$

   is a ring homomorphism.

b. Show that $\ker \psi = \langle 3, 1 + \sqrt{-5} \rangle$.

   Hint: One containment is easy if you use problem 1 from homework 1.

c. In lecture last week I claimed that 1 and -1 are not in $\langle 3, 1 + \sqrt{-5} \rangle$. Prove this by showing that they're not in $\ker \psi$.

   Hint: Don't work hard.

d. Draw a picture of R as a lattice of dots in $\mathbb{C}$, going at least 3 dots in every direction. For each point z in your lattice, color it red if $\psi(z) = \bar{0}$, green if $\psi(z) = \bar{1}$, and blue if $\psi(z) = \bar{2}$; or choose different colors according to your own taste.

(What is this about? Two points $z, w \in R$ are the same color if and only if $z \equiv w \pmod{I}$, where $I = \ker \psi = \langle 3, 1 + \sqrt{-5} \rangle$. So you're color-coding the equivalence classes mod I. In particular, the red points are I itself.)

§4.1 #8.

§4.1 #16. But some copies of the book have weird typos in this problem that make it incomprehensible, so if you think yours might have a typo check out `shifrin-123-124.pdf` in the Files section of Canvas.