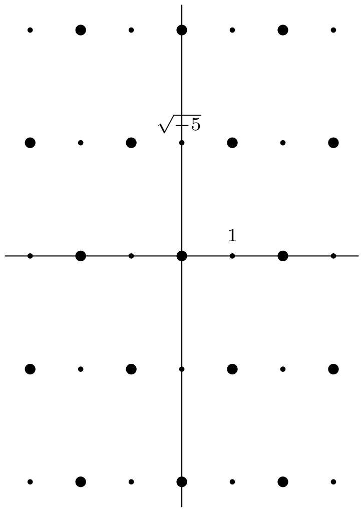
Worksheet 24

Math 391, Abstract Algebra

Friday, December 4, 2020

In lecture we studied the ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$, and the ideal $I = \langle 2, 1 + \sqrt{-5} \rangle \subset R$.



The elements of I is shown in heavier dots. On this worksheet you'll prove that I is not a principal ideal: that is, there is no $z \in R$ such that $I = \langle z \rangle$.

1. For an element $z = a + b\sqrt{-5} \in R$, we have $|z|^2 = a^2 + 5b^2$, which is an integer. Here $|\cdots|$ is the absolute value (or length) of a complex number. Convince yourselves that if z divides w in R, then $|z|^2$ divides $|w|^2$ in $\mathbb{Z}$.

   Hint: We know from Chapter 2 that $|z \cdot v| = |z| \cdot |v|$.

2. If $I = \langle z \rangle$ for some $z \in R$, then 2 and $1 + \sqrt{-5}$ would be multiples of z. Thus $|z|^2$ has to divide both 4 and 6, so $|z|^2 = 1$ or 2. Do you agree?

3. Convince yourselves that the only elements of R with $|z|^2 = 1$ are $z = \pm 1$, and there are no elements with $|z|^2 = 2$. You could do this algebraically, or (maybe better?) by drawing some circles on the grid above.

4. The ideals $\langle 1 \rangle$ and $\langle -1 \rangle$ are actually the same – what do they look like? In particular, they contain I, but they're not the same as I. So I can't be a principal ideal.