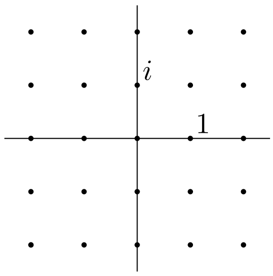
Worksheet 23

Math 391, Abstract Algebra

Monday, November 30, 2020

The Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ look like this:



1. Show that the map $\varphi\colon \mathbb{Z}[i] \to \mathbb{Z}_2$ given by $\varphi(a + bi) = \bar{a} + \bar{b}$ is a homomorphism. This means:
   (a) $\varphi(1) = \bar{1}$,
   (b) $\varphi(z + w) = \varphi(z) + \varphi(w)$ for all $z, w \in \mathbb{Z}[i]$, and
   (c) $\varphi(z \cdot w) = \varphi(z) \cdot \varphi(w)$ for all $z, w \in \mathbb{Z}[i]$.
2. Show that the map $\psi\colon \mathbb{Z}[i] \to \mathbb{Z}_2$ given by $\psi(a + bi) = \bar{a}$ is *not* a homomorphism.

   Hint: We see that $\psi(1) = \bar{1}$, so you need to find two elements $z, w \in \mathbb{Z}[i]$ such that either
   $$\psi(z + w) \neq \psi(z) + \psi(w),$$
   or
   $$\psi(z \cdot w) \neq \psi(z) \cdot \psi(w).$$
3. If you have time, take the picture above and color the points that belong to $\ker(\varphi)$, that is, the points $z \in \mathbb{Z}[i]$ with $\varphi(z) = \bar{0}$.

   Challenge: Show that this is exactly the set of all multiples of $1 + i$.