Solutions to Worksheet 10

Math 391, Abstract Algebra

This is based on Shifrin §1.4 #6 and #10.

Here are the ring axioms, copied from page 38:

(1) Addition is commutative: for all $a, b \in R$ we have $a + b = b + a$.

(2) Addition is associative: for all $a, b, c \in R$ we have $(a+b)+c = a+(b+c)$.

(3) Additive identity: There is an element $0 \in R$ such that for all $a \in R$ we have $0 + a = a$.

(4) Additive inverses: For each $a \in R$ there is an element $-a \in R$ such that $a + (-a) = 0$.

(5) Multiplication is associative: for all $a, b, c \in R$ we have $(ab)c = a(bc)$.

(6) Multiplicative unit: There is an element $1 \in R$, different from 0, such that for all $a \in R$ we have $a \cdot 1 = 1 \cdot a = a$.

(7) Distributivity: For all $a, b, c \in R$ we have $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Show that the familiar facts below follow from the ring axioms. With your colleagues, decide who will work on each one, and then present your answers to each other. Or if something gives you trouble, discuss it together.

(a) The additive identity is unique: If $0' \in R$ is another element such that for all $a \in R$ we have $0' + a = a$, then $0' = 0$.

Hint: Consider $0 + 0'$.

Solution: On the one hand,

$$0 + 0' = 0'$$

using axiom 3. On the other hand,

$$0 + 0' = 0' + 0 = 0$$

using axiom 1 and then our assumption that $0' + a = a$ for all $a \in R$.

(b) The multiplicative identity is unique: If $1' \in R$ is another element such that for all $a \in R$ we have $a \cdot 1' = 1' \cdot a = a$, then $1' = 1$.

Hint: Consider $1 \cdot 1'$.

Solution: This is similar. On the one hand,

$$1 \cdot 1' = 1'$$

using axiom 6. On the other hand,

$$1 \cdot 1' = 1$$

using our assumption that $a \cdot 1' = a$ for all $a \in R$.

(c) Additive inverses are unique: given some $a \in R$, if $b, c \in R$ satisfy $a + b = 0$ and $a + c = 0$, then $b = c$.

Hint: Consider $b + a + c$.

Solution: On the one hand,

$$(b + a) + c = (a + b) + c = 0 + c = c$$

using axiom 1, our assumption, and axiom 3. On the other hand,

$$(b + a) + c = b + (a + c) = b + 0 = 0 + b = b$$

using axiom 2, our assumption, axiom 1, and axiom 3.

(d) Multiplicative inverses are unique, when they exist: given some $a \in R$, if $b, c \in R$ satisfy $a \cdot b = b \cdot a = 1$ and $a \cdot c = c \cdot a = 1$, then $b = c$.

Hint: Consider $b \cdot a \cdot c$.

Solution: This is similar. On the one hand,

$$(b \cdot a) \cdot c = 1 \cdot c = c$$

using our assumption and axiom 6. On the other hand,

$$(b \cdot a) \cdot c = b \cdot (a \cdot c) = b \cdot 1 = b$$

using axiom 5, our assumption, axiom 1, and axiom 3.

(e) For all $a \in R$ we have $0 \cdot a = a \cdot 0 = 0$.

Hint: Simplify $(0 + 0) \cdot a$ in two ways. Add $-(0 \cdot a)$ to both sides.

Solution: On the one hand, $(0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$ using axiom 7. On the other hand, $0 + 0 = 0$ using axiom 3, so $(0+0) \cdot a = 0 \cdot a$. Thus

$$0 \cdot a + 0 \cdot a = 0 \cdot a.$$

Take $-(0 \cdot a)$, whose existence is guaranteed by axiom 4, and add it to both sides to get

$$(0 \cdot a + 0 \cdot a) + -(0 \cdot a) = 0 \cdot a + -(0 \cdot a).$$

The right-hand side is 0 by axiom 4. The left-hand side is

$$0 \cdot a + (0 \cdot a + -(0 \cdot a)) = 0 \cdot a + 0 = 0 \cdot a$$

using axiom 2, axiom 4, and axiom 3. Thus $0 \cdot a = 0$.

The proof that $a \cdot 0 = 0$ is similar, starting from $a \cdot (0 + 0)$.

(f) For all $a \in R$ we have $-(-a) = a$.

Solution: The element $-(-a)$ is defined by the fact that

$$-a + -(-a) = 0.$$

On the other hand, we have $a + -a = 0$, so $-a + a = 0$ by axiom 1. From part (c) we conclude that $-(-a) = a$.

(g) For all $a \in R$ we have $-a = (-1) \cdot a = a \cdot (-1)$.

Hint: Simplify $(1 + -1) \cdot a$ in two different ways and use part (d). [This was a typo: it should have said part (e).]

Solution: On the one hand,

$$(1 + -1) \cdot a = 1 \cdot a + (-1) \cdot a = a + (-1) \cdot a$$

using axiom 7 and axiom 6. On the other hand, $1 + -1 = 0$ so

$$(1 + -1) \cdot a = 0 \cdot a = 0$$

by part (e). Thus

$$a + (-1) \cdot a = 0,$$

so $(-1) \cdot a = -a$ by part (c).

The proof that $a \cdot 0 = 0$ is similar, starting from $a \cdot (1 + -1)$.

(h) For all $a, b \in R$ we have $-(a+b) = -a + -b$.

Solution: We have

$$\begin{aligned}
(a+b) + (-a + -b) &= (a+b) + (-b + -a) && \text{(axiom 1)}\\
&= ((a+b) + -b) + -a && \text{(axiom 2)}\\
&= (a + (b + -b)) + -a && \text{(axiom 2)}\\
&= (a+0) + -a && \text{(axiom 4)}\\
&= a + -a && \text{(axiom 3)}\\
&= 0. && \text{(axiom 4)}
\end{aligned}$$

Thus $-(a+b) = -a + -b$ by part (c).

Alternatively, you could say

$$-a + -b = (-1) \cdot a + (-1) \cdot b = (-1) \cdot (a+b) = -(a+b)$$

using part (g), axiom 7, and part (g) again.

(i) For all $a, b \in R$ we have $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.

Solution: We have

$$\begin{aligned}
(-a) \cdot b + a \cdot b &= (-a + a) \cdot b && \text{(axiom 7)}\\
&= 0 \cdot b && \text{(axiom 4)}\\
&= 0. && \text{(part (e))}
\end{aligned}$$

Thus $-(a \cdot b) = (-a) \cdot b$ by part (c).

Alternatively, you could say

$$(-a) \cdot b = ((-1) \cdot a) \cdot b = (-1) \cdot (a \cdot b) = -(a \cdot b)$$

using part (g), axiom 5, and part (g) again.

The proof that $a \cdot (-b) = -(a \cdot b)$ is similar.

(j) For all $a, b \in R$ we have $(-a) \cdot (-b) = a \cdot b$.

Solution: By part (i) we have

$$(-a) \cdot (-b) = -(a \cdot (-b)) = -(-(a \cdot b)),$$

and by part (f) this is $a \cdot b$.