Solutions to Homework 6

§2.3 #6 *Use de Moivre's theorem (Corollary 3.3) to express the following in terms of* $\sin\theta$ *and* $\cos\theta$*. The binomial theorem may prove helpful.*

First we have

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta
\end{aligned}$$

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta \\
&\qquad - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta
\end{aligned}$$

$$\begin{aligned}
\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\qquad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta
\end{aligned}$$

Equating the real and imaginary parts of the left-hand and right-hand sides in each case we get

a. $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

b. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

c. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

d. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

e. $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

§2.3 #10

a. *Use the answer to Exercise 6d to express* $\cos 4\theta$ *as a polynomial in only* $\cos\theta$*.*

From 6d we have

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

b. *Use your answer to part a to prove that* $\cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$*. Be sure to remove the* $\pm$ *ambiguity in your answer.*

From part a we have

$$8\cos^4(\pi/8) - 8\cos^2(\pi/8) + 1 = \cos(\pi/2) = 0,$$

so the quadratic formula gives

$$\cos^2(\pi/8) = \frac{8 \pm \sqrt{64-32}}{16} = \frac{2\pm\sqrt{2}}{4},$$

so

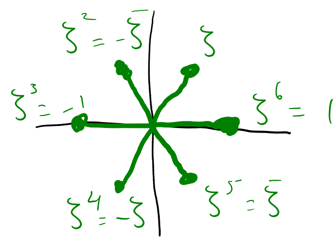
$$\cos(\pi/8) = \pm\frac{\sqrt{2\pm\sqrt{2}}}{2},$$

where the two plus-or-minuses are independent. The four choices we are getting here are the cosines of $\pi/8$, $3\pi/8$, $5\pi/8$, and $7\pi/8$. Of these, $\pi/8$ is the furthest to the right, so we want to take $+$ in both cases to get the greatest cosine.

§2.3 #15 *Use roots of unity to show that*

a. $x^6 - 1 = (x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1)$.

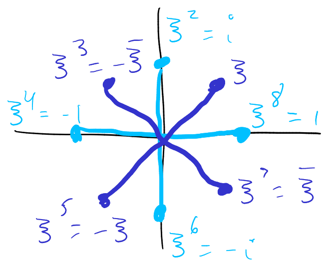
Let
$$\zeta = e^{\pi i/3} = \cos(\pi/3) + i\sin(\pi/3) = \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$
be a primitive sixth root of unity. Then the roots of $x^6 - 1$ are 1, ζ, ζ^2, ζ^3, ζ^4, and ζ^5, so
$$x^6 - 1 = (x-1)(x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5).$$
We see that $\zeta^3 = -1$, so $\zeta^4 = -\zeta$ and $\zeta^5 = -\zeta^2$. Moreover, because $\zeta^6 = 1$, we have $\zeta^5 = \zeta^{-1}$, and because $|\zeta| = 1$, we have $\zeta^{-1} = \bar{\zeta}$, and thus also $\zeta^2 = -\bar{\zeta}$. So the product can be rewritten as
$$x^6 - 1 = (x-1)(x-\zeta)(x+\bar{\zeta})(x+1)(x+\zeta)(x-\bar{\zeta}).$$



Pairing the second factor with the last we get
$$(x-\zeta)(x-\bar{\zeta}) = x^2 - (\zeta + \bar{\zeta})x + \zeta \cdot \bar{\zeta} = x^2 - x + 1,$$
and pairing the third with the fourth we get
$$(x+\zeta)(x+\bar{\zeta}) = x^2 + (\zeta + \bar{\zeta})x + \zeta \cdot \bar{\zeta} = x^2 + x + 1,$$
and hence the desired factorization.

b. $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.

Let
$$\xi = e^{\pi i/4} = \cos(\pi/4) + i\sin(\pi/4) = \tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}$$
be a primitive eighth root of unity. Then the roots of $x^4 + 1$ are ξ, ξ^3, ξ^5, and ξ^7, so
$$x^4 + 1 = (x-\xi)(x-\xi^3)(x-\xi^5)(x-\xi^7).$$
We see that $\xi^4 = -1$, so $\xi^5 = -\xi$ and $\xi^7 = -\xi^3$. Moreover, we have $\xi^7 = \xi^{-1} = \bar{\xi}$, and thus also $\xi^3 = -\bar{\xi}$. So the product can be rewritten as
$$x^4 + 1 = (x-\xi)(x+\bar{\xi})(x+\xi)(x-\bar{\xi}).$$



Pairing the first factor with the last we get
$$(x-\xi)(x-\bar{\xi}) = x^2 - (\xi + \bar{\xi})x + \xi \cdot \bar{\xi} = x^2 - \sqrt{2}x + 1,$$
and pairing the second with the third we get
$$(x+\xi)(x+\bar{\xi}) = x^2 + (\xi + \bar{\xi})x + \xi \cdot \bar{\xi} = x^2 + \sqrt{2}x + 1,$$
and hence the desired factorization.

§2.4 #1 *Solve the following equations using Theorem 4.1 (the quadratic formula).* [You got to choose two parts, but I'll do them all.]

a. $z^2 - (6+i)z + (8+2i) = 0$.

The quadratic formula gives

$$z = \frac{6 + i \pm \sqrt{3+4i}}{2}.$$

Calculating as in §2.3 Example 9, we find that $\pm\sqrt{3+4i} = \pm(2+i)$, so $z = 4+i$ or 2.

b. $z^2 - iz + 2 = 0$.

The quadratic formula gives

$$z = \frac{i \pm 3i}{2},$$

so $z = 2i$ or $-i$.

c. $z^4 + 2z^2 + 4 = 0$.

The quadratic formula gives

$$z^2 = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm i\sqrt{3},$$

which we recognize as $2e^{2\pi i/3}$ and $2e^{4\pi i/3}$. Thus $z = \pm\sqrt{2}e^{\pi i/3}$ or $\pm\sqrt{2}e^{2\pi i/3}$, that is,

$$z = \sqrt{2} \cdot \frac{\pm 1 \pm i\sqrt{3}}{2},$$

where the two plus-or-minuses are independent.

d. $(x^2 + 3x + 4)(x^2 + 3x + 5) = 6$.

Since $6 = 2 \cdot 3$, we guess, wildly, that $x^2 + 3x + 4 = 2$ and $x^2 + 3x + 5 = 3$. We get lucky, and see that $x = -1$ and $x = -2$ are solutions. Then we take

$$(x^2 + 3x + 4)(x^2 + 3x + 5) - 6 = x^4 + 6x^3 + 18x^2 + 27x + 14$$

and divide it by $x + 1$ and $x + 2$ to get

$$x^2 + 3x + 7,$$

whose two roots are

$$x = \frac{-3 \pm i\sqrt{19}}{2}.$$

e. $2x^4 - 3z^3 + 4z^2 - 3z + 2 = 0$. (Hint: Divide by z^2.)

Following the hint, we observe that $z = 0$ is not a root of the quartic, so we can divide through by z^2 to get

$$2z^2 - 3z + 4 - \frac{3}{z} + \frac{2}{z^2} = 0.$$

We reorder the terms to get

$$2z^2 + 4 + \frac{2}{z^2} - 3z - \frac{3}{z} = 0,$$

and factor this to get

$$2\left(z + \frac{1}{z}\right)^2 - 3\left(z + \frac{1}{z}\right) = 0,$$

so either $z + \frac{1}{z} = 0$ or $2(z + \frac{1}{z}) - 3 = 0$. Multiplying back through by z, the first becomes $z^2 + 1 = 0$, so $z = \pm i$, and the second becomes

$$2z^2 - 3z + 2 = 0,$$

so

$$z = \frac{3 \pm i\sqrt{7}}{4}.$$

f. $16x^4 - 20x^2 + 5 = 0$.

The quadratic formula gives

$$x^2 = \frac{20 \pm \sqrt{80}}{32} = \frac{5 \pm \sqrt{5}}{8},$$

so

$$x = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}},$$

where the two plus-or-minuses are independent.

§2.4 #6 *Solve the following cubic equations.*

Throughout we let $\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be a primitive cube root of unity, so $\omega^2 = \omega^{-1} = \bar{\omega} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ is the other one.

a. $z^3 - 9z - 28 = 0$.

In the notation of Theorem 4.3 we can take

$$A = 14 + \sqrt{169} = 27.$$

The three cube roots of A are 3, 3ω, and $3\omega^2$, so our three solutions are

$$\begin{aligned} z &= 3 + 1 = 4 \\ z &= 3\omega + \omega^{-1} = 3\omega + \bar{\omega} = -2 + i\sqrt{3} \\ z &= 3\omega^2 + \omega^{-2} = 3\bar{\omega} + \omega = -2 - i\sqrt{3}. \end{aligned}$$

b. We substitute $z = y + 3$, so the equation becomes

$$y^3 - 18y - 35.$$

In the notation of Theorem 4.3 we can take

$$A = \tfrac{35}{2} + \sqrt{\tfrac{361}{4}} = 27,$$

so again the three cube roots of A are 3, 3ω, and $3\omega^2$. Then we get

$$y = \begin{cases} 3 + 2 = 5 \\ 3\omega + 2\omega^{-1} = 3\omega + 2\bar{\omega} = \frac{-5}{2} + i\frac{\sqrt{3}}{2} \\ 3\omega^2 + 2\omega^{-2} = 3\bar{\omega} + 2\omega = \frac{-5}{2} - i\frac{\sqrt{3}}{2}, \end{cases}$$

so

$$z = y + 3 = \begin{cases} 8 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{1-i\sqrt{3}}{2}. \end{cases}$$

c. $z^3 - 3z - 1 = 0$.

In the notation of Theorem 4.3 we can take

$$A = \tfrac{1}{2} + \sqrt{-\tfrac{3}{4}} = \frac{1 + \sqrt{-3}}{2} = e^{\pi i/3}.$$

Let $\zeta = e^{\pi i/9} = \cos 20^\circ + i \sin 20^\circ$. Then the three cube roots of A are ζ,

$$\zeta \cdot \omega = \zeta^7 = \cos 140^\circ + i \sin 140^\circ,$$

and

$$\zeta \cdot \omega^2 = \zeta^{13} = \cos 260^\circ + i \sin 260^\circ.$$

Thus our three solutions are

$$z = \begin{cases} \zeta + \zeta^{-1} = \zeta + \bar{\zeta} = 2\cos 20^\circ \\ \zeta^7 + \zeta^{-7} = \zeta^7 + \bar{\zeta}^7 = 2\cos 140^\circ \\ \zeta^{13} + \zeta^{-13} = \zeta^{13} + \bar{\zeta}^{13} = 2\cos 260^\circ. \end{cases}$$