Solutions to Homework 4

§1.4 #11. We have a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries, and we let $\Delta = \det A = ad - bc$.

a. *Show that A is a unit in $M_2(\mathbb{Z})$ if and only if $\Delta = \pm 1$.*

   If $\Delta = 1$, we find that

   $$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   satisfies $AB = BA = 1$, so A is a unit and B is its inverse. Similarly, if $\Delta = -1$ then

   $$\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

   is inverse to A.

   Conversely, suppose that A is a unit. By definition, this means there is a $C \in M_2(\mathbb{Z})$ such that $AC = CA = 1$. Taking determinants we find that $\det A \cdot \det C = 1$. But $\det A$ and $\det C$ are integers, so we must have $\det A = \det C = \pm 1$.

b. *Show that A is a a zero-divisor in $M_2(\mathbb{Z})$ if and only if* $\det A = 0$ *and not all of a, b, c, and d are zero.*

If $\det A = 0$, then we find that

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is a non-zero matrix satisfying $AB = 0$.

Conversely, suppose that A is a zero-divisor. By definition, this means there is a $C \in M_2(\mathbb{Z})$ such that $AC = 0$ or $CA = 0$. Write

$$C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

If $AC = 0$, we find that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is, the columns of C are in the null space of A (also known as the kernel of A). Because $C \neq 0$, one of those columns must be non-zero, so the null space is non-trivial, so $\det A = 0$.

Similarly, if $CA = 0$ then taking rows of C we find that the left null space of A is non-trivial, so $\det A = 0$. Or if you prefer, take transposes to get $A^\top C^\top = 0$, so $\det(A^\top) = 0$, but $\det(A^\top) = \det A$.

For a fancier approach to the converse, you could again let

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and observe that

$$AB = BA = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}.$$

Now if there is a non-zero C such that $AC = 0$ or $CA = 0$, then multiplying by B we get $0 = BAC = \Delta \cdot C$ or $0 = CAB = C \cdot \Delta$. Because some entry of C is non-zero, we must have $\Delta = 0$.

§1.4 #4. *Which of the following matrices are units in $M_2(\mathbb{Z})$? Which are zero-divisors? Which are neither?*

a. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

   The determinant is $2 \cdot 3 - 5 \cdot 1 = 1$, so it is a unit.

   The inverse is $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

b. $\begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix}$

   The determinant is $3 \cdot 2 - 1 \cdot 7 = -1$, so it is a unit.

   The inverse is $\begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix}$.

c. $\begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$

   The determinant is $2 \cdot 4 - 5 \cdot 1 = 3$, so it is neither a unit nor a zero-divisor.

d. $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

   The determinant is $2 \cdot 2 - 4 \cdot 1 = 0$, so it is a zero-divisor. For example, if we multiply on either side with $\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}$ then we get zero.

e. $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$. The determinant is $0 \cdot (-1) - 0 \cdot 1 = 0$, so it is a zero-divisor.

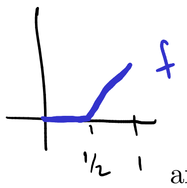
   For example, if we multiply on either side with $\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ then we get zero.

§1.4 #14.

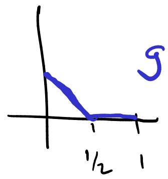
b. *Let R be the set of all real-valued functions on the closed interval $[0,1]$. Is R an integral domain? What are the units and zero-divisors? Is every non-zero element of R either a unit or a zero-divisor?*

Observe that R is commutative, that the constant function 0 is the additive identity in R, and the constant function 1 is the multiplicative identity.

No, R is not an integral domain: for example the functions



$$f(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} & \frac{1}{2} \le x \le 1 \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} - x & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases}$$

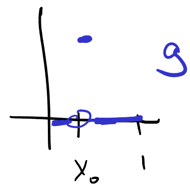


are non-zero, but $fg = 0$.

First I claim that an element $f \in R$ is a unit if and only if it never vanishes, that is, $f(x) \neq 0$ for all $x \in [0,1]$. In one direction, suppose that f never vanishes, and define $g(x) = 1/f(x)$. Then $fg = 1$, so f is a unit. Conversely, suppose that there is a $g \in R$ such that $fg = 1$. Then for every $x \in [0,1]$ we have $f(x)g(x) = 1$, so $f(x) \neq 0$.

Next I claim that a non-zero element $f \in R$ is a zero-divisor if and only if it vanishes at some point, i.e. there is an $x_0 \in [0,1]$ such that $f(x_0) = 0$. In one direction, suppose there is such an x_0, and define

$$g(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0. \end{cases}$$



Then $g \neq 0$ but $fg = 0$, so f is a zero-divisor. Conversely, suppose there is a $g \neq 0$ such that $fg = 0$. Since $g \neq 0$, there is a point $x_0 \in [0,1]$ such that $g(x_0) \neq 0$. Since $fg = 0$, we have $f(x_0)g(x_0) = 0$, so $f(x_0) = 0$.

In particular we see that any non-zero $f \in R$ is either a unit or a zero-divisor.

d. *Let $R' \subset R$ be the ring of* continuous *functions. Is R' an integral domain? What are the units and zero-divisors? Is every non-zero element of R' either a unit or a zero-divisor?*

No, R' is not an integral domain, using the same example as before. And it's still true that f is a unit if and only if it never vanishes, because if f is continuous and never vanishes then $1/f$ is also continuous.

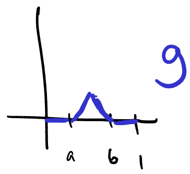
But the zero-divisors are different, and it is no longer true that every non-zero element is either a unit or a zero-divisor. For example, the function $f(x) = x - \frac{1}{2}$ is not zero, and it is not a unit because $f(1/2) = 0$, but it is not a zero-divisor either: if $fg = 0$ then $g(x) = 0$ for all $x \neq 1/2$, and because g is continuous we have

$$g(1/2) = \lim_{x \to 1/2} g(x) = 0,$$

so $g = 0$.

I claim that a non-zero element $f \in R'$ is a zero-divisor if and only if it vanishes on some open interval (a, b), $0 \leq a < b \leq 1$. In one direction, suppose that there is such an open interval, and define

$$g(x) = \begin{cases} 0 & x \leq a \\ x - a & a \leq x \leq \frac{a+b}{2} \\ b - x & \frac{a+b}{2} \leq x \leq b \\ 0 & x \leq b. \end{cases}$$



Then g is continuous, and $g(\frac{a+b}{2}) = \frac{b-a}{2} \neq 0$, so $g \neq 0$, but $fg = 0$. Conversely, suppose there is a continuous $g \neq 0$ such that $fg = 0$. Because $g \neq 0$, there is a point $x_0 \in [0, 1]$ such that $g(x_0) \neq 0$. Because g is continuous, there is $\delta > 0$ such that $g(x) \neq 0$ for whenever $x \in (x_0 - \delta, x_0 + \delta)$.[1] Because $fg = 0$, we must have $f(x) = 0$ for x in the same interval.[2]

[1] You can see this by letting $\epsilon = |g(x)|$ in the definition of continuity.

[2] If you wanted to be really picky, you could explain that $(x_0 - \delta, x_0 + \delta) \cap [0, 1]$ contains the open interval (a, b), where $a = \max(0, x_0 - \delta)$ and $b = \min(1, x + \delta)$.

The problem that's not from the book.
Let R be a commutative ring, fix an element $r \in R$, and consider the map $R \to R$ given by $x \mapsto rx$.

Let's call the map φ for definiteness, so $\varphi(x) = rx$.

Show that it is injective if and only if r is neither zero nor a zero-divisor,

I'll show that φ is *not* injective if and only if r is either zero or a zero-divisor, which is equivalent.

If φ is not injective, then there are two elements $x, y \in R$ with $x \neq y$ but $\varphi(x) = \varphi(y)$, so $rx = ry$, so $r(x - y) = 0$. Let $s = x - y$, so $s \neq 0$ but $rs = 0$; thus either r is zero or r is a zero-divisor.

Conversely, if $r = 0$ then $\varphi(0) = 0 \cdot 0 = 0$, and $\varphi(1) = 1 \cdot 0 = 0$, and the ring axioms include $1 \neq 0$, so φ is not injective. And if r is a zero-divisor then there is some $s \in R$ with $s \neq 0$ and $rs = 0$, so $\varphi(s) = 0$, but we still have $\varphi(0) = 0$, so again φ is not injective.

and that it is surjective if and only if r is a unit.

If φ is surjective then there is an $x \in R$ with $\varphi(x) = 1$, that is, $rx = 1$, so r is a unit. Conversely, if r is a unit then there is an $s \in R$ with $rs = 1$. For any $x \in R$ we have $rsx = x$, so $\varphi(sx) = x$. Thus φ is surjective.