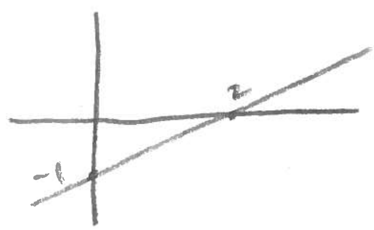
Solutions to Midterm 1

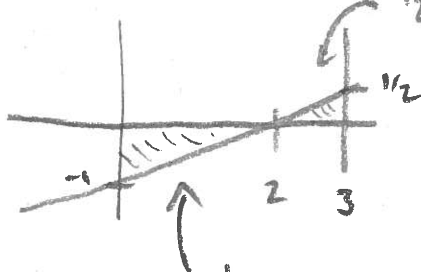
1. §5.2 #33:

   (a) Sketch the line $y = \frac{1}{2}x - 1$.
   Solution:



   (b) Find $\int_0^3 (\frac{1}{2}x - 1)\,dx$ geometrically, using triangles.
   Solution:

$\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$



$-1 + \frac{1}{4} = -\frac{3}{4}$

$\frac{1}{2} \cdot 2 \cdot 1 = 1$

   (c) Find $\int_0^3 (\frac{1}{2}x - 1)\,dx$ in the usual algebraic way, using anti-derivatives.
   Solution:

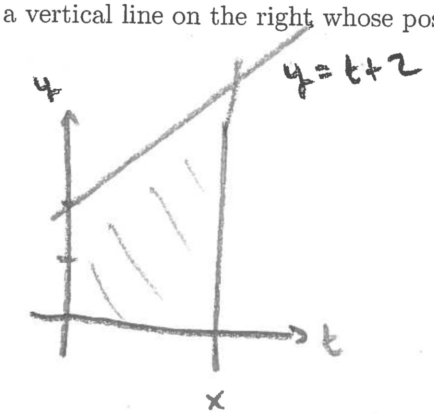
$$\int_0^3 (\tfrac{1}{2}x - 1)\,dx = \left[\tfrac{1}{4}x^2 - x\right]_0^3 = (\tfrac{9}{4} - 3) - (0 - 0) = -\tfrac{3}{4}$$

2. This problem is about the fundamental theorem of calculus.

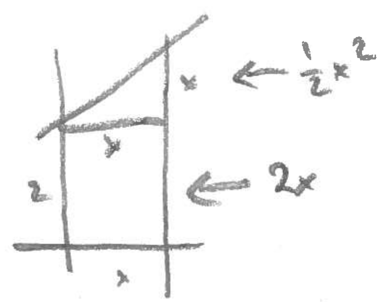
For a positive number x, let

$$F(x) = \int_0^x (t+2)\,dt.$$

(a) Sketch the region whose area is given by $F(x)$.
Hint: There is a vertical line on the right whose position is x.
Solution:



(b) Find $F(x)$ geometrically, using a triangle and a rectangle.
Solution:



$$F(x) = \tfrac{1}{2}x^2 + 2x.$$

(c) Find $F'(x)$.
Solution:

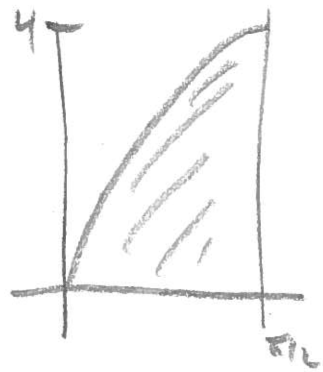
$$\tfrac{d}{dx}\left(\tfrac{1}{2}x^2 + 2x\right) = x + 2.$$

3. Find

$$\int_0^{\pi/2} 4\sin x\,dx.$$

Bonus (2 points): Sketch the region whose area this integral computes.
Solution:

$$\int_0^{\pi/2} 4\sin x\,dx = -4\cos x\Big|_0^{\pi/2} = 0-(-4) = 4.$$



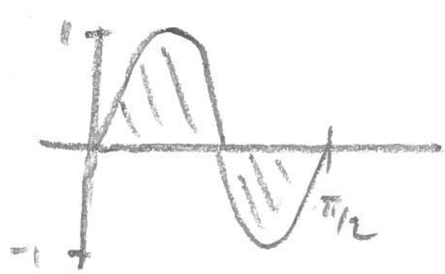
4. Find

$$\int_0^{\pi/2} \sin 4x\,dx.$$

Bonus (2 points): Sketch the region whose area this integral computes.

Solution: Substitute $u = 4x$, so $\frac{du}{dx} = 4$, so $\frac{1}{4}du = dx$.
If $x = 0$ then $u = 0$. If $x = \pi/2$ then $u = 2\pi$. Thus

$$\int_0^{\pi/2} \sin x\,dx = \int_0^{2\pi} \sin u \cdot \tfrac{1}{4}\,du = -\tfrac{1}{4}\cos u\Big|_0^{2\pi} = -\frac{1}{4}(1-1) = 0.$$



5. Based on §5.5 #43: Find

$$\int_0^1 x\sqrt[3]{1+7x^2}\,dx.$$

Hint: Substitute $u = 1 + 7x^2$.

To clean up at the end, recall that $8^{1/3} = \sqrt[3]{8} = 2$.

Solution: Letting $u = 1 + 7x^2$, we have $\frac{du}{dx} = 14x$, so $\frac{1}{14}du = x\,dx$. If $x = 0$ then $u = 1$. If $x = 1$ then $u = 8$. Thus

$$\begin{aligned}
\int_0^1 x\sqrt[3]{1+7x^2}\,dx &= \int_1^8 \sqrt[3]{u}\cdot\tfrac{1}{14}\,du \\
&= \tfrac{1}{14}\int_1^8 u^{1/3}\,du \\
&= \tfrac{1}{14}\cdot\tfrac{3}{4}u^{4/3}\Big|_1^8 \\
&= \tfrac{3}{56}(2^4-1) = \tfrac{3}{56}(15) = \tfrac{45}{56}
\end{aligned}$$

6. Find

$$\int_0^\pi x\cos x\,dx.$$

Hint: Integrate by parts, letting $u = x$ and $v' = \cos x$.

Solution: Then $u' = 1$ and $v = \sin x$, so

$$\begin{aligned}
\int_0^\pi x\cos x\,dx &= x\sin x\Big|_0^\pi - \int_0^\pi \sin x\,dx \\
&= \pi\cdot 0 - 0\cdot 0 - \big[-\cos x\big]_0^\pi \\
&= \Big[\cos x\Big]_0^\pi \\
&= -1-1 = -2.
\end{aligned}$$

7. Find

$$\int_1^2 x \ln x \, dx.$$

Hint: Integrate by parts, letting $u = \ln x$ and $v' = x$.

To clean up at the end, recall that $\ln 1 = 0$.

Solution: Then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$, so

$$\begin{aligned}\int_1^2 x \ln x \, dx &= \frac{1}{2}x^2 \ln x \Big|_1^2 - \int_1^2 \tfrac{1}{2}x \, dx \\ &= 2\ln 2 - 0 - \left[\tfrac{1}{4}x^2\right]_1^2 \\ &= 2\ln 2 - (1 - \tfrac{1}{4}) \\ &= 2\ln 2 - \tfrac{3}{4}.\end{aligned}$$

8. Find

$$\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx.$$

Hint: Use the identity $\cos^2 x = 1 - \sin^2 x$. Substitute $u = \sin x$.

Solution: First,

$$\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^2 x (1 - \sin^2 x) \cos x \, dx.$$

We let $u = \sin x$, so $\frac{du}{dx} = \cos x$, so $du = \cos x \, dx$. If $x = 0$ then $u = 0$. If $x = \pi/2$ then $u = 1$. Thus the integral becomes

$$\begin{aligned}\int_0^1 u^2(1 - u^2) \, du &= \int_0^1 (u^2 - u^4) \, du \\ &= \left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]_0^1 \\ &= \tfrac{1}{3} - \tfrac{1}{5} - 0 = \tfrac{2}{15}.\end{aligned}$$