Final Exam

Name ____

Do problems 1 and 2, and **any two** of problems 3 to 6. Each problem is worth 20 points total. You will not get extra credit for doing more than two differential equations problems. You may use a scientific calculator but not a graphing calculator.

1. (a) Evaluate the *indefinite* integral

$$\int \frac{1}{(2x-1)^2}\,dx.$$

   (b) Take the derivative of your answer to part (a), using the chain rule. Is it $\frac{1}{(2x-1)^2}$? If not, revise your answer to part (a).

(c) Evalutate

$$\int_1^\infty \frac{1}{(2x-1)^2}\,dx.$$

(d) Extra credit: Sketch the graph of $y = \frac{1}{(2x-1)^2}$. Then evalutate

$$\int_0^1 \frac{1}{(2x-1)^2}\,dx.$$

2. On the board you see a sketch of the graph $y = x^{3/2}$.

   (a) Set up an integral (but do not evaluate it!) to find the length of the part of the curve that lies between the points $(0, 0)$ and $(1, 1)$.

   (b) Evaluate the integral you set up in part (a).
   Hint: It should be an easy u-substitution, not anything crazy. At the end it's probably better to get a decimal approximation than to spend time simplifying.

   (c) Extra credit: Draw a picture to explain why your answer should be between $\sqrt{2} \approx 1.4142$ and 2. Check that your answer lies in this range. If not, revise your answers to (a) and (b).

Do **any two** of problems 3–6.

3. You iron a cotton shirt using a iron heated to 400°F. You switch off the iron and leave it to cool in a room whose temperature is 70°F. After three minutes, the iron has cooled to 200°F.

    (a) Let y denote the temperature of the iron in degrees Fahrenheit, and let t denote the time in minutes. Write an equation involving dy/dt that expresses Newton's law of cooling: the rate at which the iron cools is proportional to the difference between the termperature of the iron and the termperature of the room.

    (b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

    (c) Plug in the data to determine the constants in part (b).

(d) What will be the temperature of the iron one minute later – that is, four minutes after you switched it off?

(e) You decide to put the iron away in the closet when it cools down to 100°F. At what time will that happen?

Do **any two** of problems 3–6.

4. Based on §7.5 #10.

   The department of fish and wildlife stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year.

   (a) Let y denote the number of fish in the lake, and let t denote the time in years. Assume a logistic growth model: that is, the rate at which the population grows is proportional to both the number of fish *and* how far the number is below the carrying capacity. Write an equation involving dy/dt to express this.

   (b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

(c) Plug in the data to determine the constants in part (b).

(d) How long will it take for the population to increase to 5,000?

(e) Due to a miscommunication with another agency, more fish were released into the lake, bringing the population to an unsustainable 15,000. How long will it take for the population to fall to 12,000?

Hint: Use the same differential equation and the same constant of proportionality k that you found earlier, but find a new C to reflect the new situation. Be careful with signs and absolute values.

Do **any two** of problems 3–6.

5. Based on §7.3 #44 or last Friday's quiz.

   A certain small country has \$10 billion in paper currency in circulation. Each day, \$50 million comes into the country's banks, and the same amount goes back out into circulation. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks.

   (a) Let x denote the amount of new currency in circulation, in millions of dollars, and let t denote the time in days after the new currency started being introduced. Write an equation involving dx/dt to model the situation described above.

   Hint: This is like a "salt in a tank" problem.

   (b) Separate variables and integrate to get an equation involving x, t, and a constant. You can solve for x if you want, but you don't have to.

   (c) Plug in the data to determine the constant in part (b).

(d) How much new currency is in circulation after 30 days?

(e) How long will it take for the new bills to account for 90% of the currency in circulation?

Do **any two** of problems 3–6.

6. According to Toricelli's law, the rate at which water drains from a bath tub is proportional to the *square root* of the depth of the water. A rectangular bath tub was initially filled to a depth of 12 inches, and took 6 minutes to drain. (You can make up a width and length for the tub but it doesn't affect the answer.)

   (a) Let y denote the depth of the water, and let t denote the time in minutes. Write an equation involving dy/dt to model the situation described above.

   (b) Separate variables and integrate to get an equation involving y, t, and some constants. You can solve for y if you want, but you don't have to.

(c) Plug in the data to determine the constant in part (b).

(d) At what time was the bath tub half full?

Sanity check: it should be earlier than 3 minutes, because the first half drains faster than the second half.