Optional problem for homework 1. Evaluate

$$\int_0^1 e^x \, dx$$

by hand, without using the fundamental theorem of calculus, as follows:

(a) Divide the interval from 0 to 1 into n equally-sized intervals. Write the left-endpoint Riemann sum as

$$\tfrac{1}{n} \cdot 1 + \tfrac{1}{n} e^{1/n} + \tfrac{1}{n} e^{2/n} + \cdots + \text{(figure out how this should end)}.$$

Draw a picture with $n = 5$.

(b) Recall the sum of the geometric series:*

$$1 + r + r^2 + \cdots + r^k = \frac{r^{k+1} - 1}{r - 1}. \qquad (*)$$

Apply this with $r = e^{1/n}$ and an appropriate k to simplify your answer to part (a).

(c) Evaluate

$$\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n}.$$

Hint: You can use l'Hôpital's rule if you want, but for a more basic approach you can let $h = 1/n$, so as $n \to \infty$ we have $h \to 0$; then recognize $\lim_{h\to 0} \frac{e^h - 1}{h}$ as the derivative of e^x at $x = 0$.

(d) Use your answer to part (c) to evaluate the limit of your answer to part (b) as $n \to \infty$.

(e) As a sanity check, evaluate $\int_0^1 e^x \, dx$ using the fundamental theorem of calculus.

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*If you want to be reminded of where this formula comes from, let $S = 1+r+r^2+\cdots+r^k$, so $rS = r+r^2+r^3+\cdots+r^{k+1}$, so if we take $rS-S$ then most of the terms cancel and we are left with $r^{k+1}-1$. Solving the equation $rS-S = r^{k+1}-1$ for S gives us the equation labeled $(*)$ above. If we take $r = 10$ then equation $(*)$ says something very familiar: for example, if $k = 3$ then we're talking about the fact that $\frac{10000-1}{10-1} = \frac{9999}{9} = 1 + 10 + 100 + 1000$.