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RATIONAL MODEL OF SUBSPACE COMPLEMENT ON ATOMIC COMPLEX

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1. Introduction

In [12], a rational model of the complement $C(X)$ to a complex subspace arrangement X is constructed that uses as the underlying complex the direct sum of the order (flag) complexes for all the intervals $[0, A]$ of the intersection lattice $L(X)$ of X. The product in this model is defined via a messy algorithm involving the shuffle product of flags (see section 3). The atomic complex of a lattice is typically much smaller than its order complex although homotopy equivalent to the latter. Thus the sum of atomic complexes of the intervals of $L(X)$ could be used potentially for computation of the algebra $H^*(C(X), \mathbb{Q})$ instead of the order complexes. However there is no natural multiplication on this sum that induces the right multiplication on homology.

In this note, we show how to overcome this difficulty by using the sum of the *relative* atomic complexes. The relative atomic complex of an interval $[0, A]$ can be interpreted as the factor complex of the simplex on all the atoms under A over the atomic complex of $[0, A]$. On the sum of these complexes, the needed multiplication is given by a very simple and natural formula. Roughly speaking, the product of two sets of atoms is either 0 or the union of these sets up to a sign. In cases where generators of local homology of $L(X)$ can be given explicitly one can use this formula and try to get a presentation of the ring $H^*(C(X), \mathbb{Q})$. We briefly consider the simplest case of so called *homologically geometric* lattices that covers geometric lattices and lattices of types $\Pi_{n,k}$. The most important unsolved problems (see section 4) include the question of naturality and (related) question of integer coefficients. I include a conjecture about the latter problem (Conjecture 3.3) that was proved recently in two particular cases in [7] and [6].

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2. Relative atomic complex of a lattice

In this section, L is an arbitrary lattice. The set of atoms of L is denoted by $\mathcal{A}(L)$ of just $\mathcal{A}$ if L is fixed. We fix an arbitrary linear order on $\mathcal{A}(L)$ and all the subsets of this set will be provided with the induced order. The smallest and largest elements of any lattice will be denoted by 0 and 1 respectively unless more specific symbols are appropriate. For every $A \in L$ put $[0, A] = \{B \in L | B \leq A\}$, $\mathcal{A}(A) = \mathcal{A}([0, A])$ and for every $\sigma \subset \mathcal{A}$ put $\bigvee(\sigma) = \bigvee_{A \in \sigma} A$.

The homology of an arbitrary poset P is the homology of its order complex $F(P)$ (i.e., the complex of flags). For a lattice L, it is customary to mean by its homology $(H_*(L))$ the homology of the poset $L_0 = L \backslash \{0, 1\}$. This homology can be computed also as the homology of the atomic complex $\Delta(L)$ on $\mathcal{A}$ whose simplexes are the subsets $\sigma \subset \mathcal{A}$ with $\bigvee(\sigma) < 1$. In fact $\Delta(L)$ is homotopy equivalent to $F(L_0)$ (see [4] and Lemma 2.1 below). For any simplicial complex Δ we denote by $C(\Delta)$ its chain complex over $\mathbb{Z}$.

We need to fix a particular homotopy equivalence of the chain complexes $C(\Delta(L))$ and $C(F(L_0))$. For that denote by $\Delta'(L)$ the barycentric subdivision of $\Delta(L)$ and identify $\Delta'(L)$ with $F(Q)$ where Q is the poset of all nonempty simplexes of $\Delta(L)$ ordered by inclusion. Denote by β the standard homotopy equivalence $\beta : C(\Delta(L)) \to C(\Delta'(L))$ defined by

$$\beta(\{A_1, \dots, A_p\}) = \sum_{\delta} (-1)^{\mathrm{sign}\delta} (\{A_{\delta(1)}\}, \{A_{\delta(1)}, A_{\delta(2)}\}, \cdots, \{A_{\delta(1)}, \dots, A_{\delta(p)}\})$$

where $\{A_1, \dots, A_p\}$ is an arbitrary simplex of $\Delta(L)$ and the summation is taken over all the permutations δ of rank p . Also define the order preserving map $\gamma : Q \to L_0$ via $\gamma(\sigma) = \bigvee(\sigma)$ for $\sigma \in \Delta(L)$ and keep the same symbol for the respective chain map $C(\Delta'(L)) = C(F(Q)) \to C(F(L_0))$. The map γ is a homotopy equivalence since for every $C \in L_0$ the poset $\gamma^{-1}(C)$ has the unique maximal element $\{A \in \mathcal{A}(L) | A \leq C\}$ (e.g., see [10]). Thus we can register the following lemma.

Lemma 2.1. *For a lattice L the group homomorphism $f_L = \gamma\beta : C(\Delta(L)) \to C(F(L_0))$ given on atomic simplexes by*

$$f(A_1, \dots, A_p) = \sum_{\delta} (-1)^{\mathrm{sign}\delta} (A_{\delta(1)} \leq A_{\delta(1)} \vee A_{\delta(2)} \leq \cdots \leq \bigvee_{i=1}^{p} A_{\delta(i)}),$$

where the summation is taken over all the permutations δ of rank p and every flag with repetitions considered to be 0, is a chain homotopy equivalence.

Now we define the relative atomic complex.

Definition 2.2. *The relative atomic (chain) complex $D = D(L)$ is the free Abelian group on all subset $\sigma = \{A_{i_1}, \dots, A_{i_p}\}$ of $\mathcal{A}$ with its differential d defined via*

$$d(\sigma) = \sum (-1)^j \sigma \setminus \{A_{i_j}\}$$

where the summation is taken over indexes j such that $\bigvee(\sigma \setminus \{A_{i_j}\}) = \bigvee(\sigma)$.

The relative atomic complex was used in [11] for computation of the homology groups of a subspace complement.

Complex D can be naturally represented as the direct sum of complexes. Indeed let $A \in L$ and denote by $\Sigma(A)$ the simplicial complex whose simplexes are all the subsets of $\mathcal{A}(A)$. Clearly $\Sigma(A)$ is just a natural simplicial decomposition of the simplex on atoms under A. Put $\Delta(A) = \Delta([0, A])$ and $\overline{\Delta(A)} = \Sigma(A))/\Delta(A)$ (using the usual agreement that contracting the empty subcomplex means adding a disjoint point). Now put $D(A) = C(\overline{\Delta(A)})$. In other words, $D(A)$ has a basis consisting of all the subsets σ of $\mathcal{A}(A)$ such that $\bigvee(\sigma) = A$ and its differential omits all $\tau \subset \sigma$ with $\bigvee(\tau) \neq A$. The following lemma is immediate.

Lemma 2.3. *(i) The complex $\overline{\Delta(A)}$ is homotopy equivalent to the suspension of $\Delta(A)$. In particular we have an isomorphism*

$$\tilde{H}_p(D(A)) \approx \tilde{H}_{p-1}(\Delta(A))$$

for every p and $A \in L$.

(ii) $D = \oplus_{A \in L} D(A)$.

Lemmas 2.1 and 2.3 imply that the homology groups of D are isomorphic to the sum of local homology of L (or Whitney homology, cf. [1]).

Now we define the binary operation on D (to be used in the next section) via

$$\sigma\tau = (-1)^{\operatorname{sign}\epsilon(\sigma,\tau)} \sigma \cup \tau$$

if $\sigma \cap \tau = \emptyset$ and

$$\sigma\tau = 0$$

otherwise. Here $\epsilon(\sigma, \tau)$ is the shuffle of $\sigma \cup \tau$ that puts all elements of τ after elements of σ.

3. Rational cohomology algebra of subspace complement

In this section we define a differential graded algebra (DGA) over $\mathbb{Q}$ whose cohomology ring is isomorphic to the cohomology ring of the complement of a complex subspace arrangement.

Let V be a finite dimensional complex linear space and X a finite set of nonzero linear subspaces of V without any inclusions among them. We are interested in the topological space $C(X) = V^* \setminus \bigcup_{A \in X} A^\circ$ where A° is the annihilator of A in V^*. This space does not change if we substitute the collection of sums of all nonempty subsets of X for X. Completing this collection by 0 and ordering it by inclusion we obtain a lattice $L = L(X)$. This lattice is labeled by the dimensions of its elements.

A rational DGA whose cohomology ring is isomorphic to $H^*(C(X),\mathbb{Q})$ was constructed by De Concini and Procesi in [5] who specialized the general construction of Morgan [9]. Then in [12], it was simplified to a DGA whose underlying complex is the sum of order complexes for all intervals $(0,A)$ $(A\in L)$. Let us recall the multiplication of this algebra.

Let $A,B\in L$ and $F=(A_1<A_2<\cdots<A_p)$ and $G=(B_1<B_2<\cdots<B_q)$ be flags (i.e., linearly ordered subsets) in $(0,A)$ and $(0,B)$ respectively. Denote by $\bar{F}$ and $\bar{G}$ the flags augmented by A and B respectively. Then for every $(p+1,q+1)$-shuffle π of $\bar{F}\cup\bar{G}$ (i.e., a permutation preserving the ordering among elements of each flag) put

$$(\bar{F}\cup\bar{G})^\pi=\lambda\pi(\bar{F}\cup\bar{G})$$

where λ is an operation creating a flag from an ordered subset of L via

$$\lambda(C_1,C_2,\dots,C_r)=(C_1,C_1+C_2,\dots,C_1+C_2+\cdots+C_r).$$

Notice that for every shuffle π the flag $H=(\bar{F}\cup\bar{G})^\pi$ has $A+B$ as his last element and denote by $\underline{H}$ the same flag with $A+B$ deleted. We keep the same symbol for this operation extended by linearity to linear combinations of flags ended on $A+B$. As usual flags with repetitions are equated to 0. Finally we define the product of F and G as

$$F\circ G=\sum_\pi(\mathrm{sign}\pi)\underline{(\bar{F}\cup\bar{G})^\pi}$$

where π runs through all the shuffles if $A\cap B=0$ (equivalently $\dim(A+B)=\dim A+\dim B$) and

$$F\circ G=0$$

otherwise.

Now we use the relative atomic complex of L to simplify this DGA further. Let D be the complex from the previous section but with coefficients in $\mathbb{Q}$ and augmented in degree -1 by a copy of $\mathbb{Q}$ (or rather a one-dimensional space with $\emptyset$ as a basis). This complex has two gradings defined on sets of atoms σ by $|\sigma|-1$ and $\dim\bigvee(\sigma)$ respectively. We define the new grading on it putting $\deg\sigma=2\dim\bigvee(\sigma)-|\sigma|$ and denote the new graded linear space by $\tilde{D}$. In particular $\tilde{D}_0$ is generated by $\emptyset$.

We want to use the product from the previous section but first need to adjust it to the new grading. Define the binary operation on $\tilde{D}$ via

$$\sigma\cdot\tau=\sigma\tau$$

if $\dim(\bigvee(\sigma\cup\tau))=\dim(\bigvee(\sigma))+\dim(\bigvee(\tau))$ and

$$\sigma\cdot\tau=0$$

otherwise.

The check of the following lemma is straightforward although tedious and left to the reader.

Lemma 3.1. *The space $\tilde{D}$ provided with the differential d of D and the product $\cdot$ is a differential graded algebra.*

The following theorem is the main result of the paper.

Theorem 3.2. *There is an algebra isomorphism $H^*(\tilde{D}) \to H^*(C(X), \mathbb{Q})$.*

Proof. The long exact homology sequence of the pair $(\Sigma(A), \Delta(A))$ gives the natural isomorphisms $\delta : \tilde{H}_p(D(A)) \to \tilde{H}_{p-1}(\Delta(A))$ for every $A \in X$ that induces a linear isomorphism $\phi : H^*(\tilde{D}) \to H^*(C(X), \mathbb{Q})$. Notice that there exist natural homomorphisms $\bar{\delta} : D_p \to C_{p-1}(\Delta(A))$ that induce δ (for every p and A). Indeed it suffices to define $\bar{\delta}$ via $\bar{\delta}(\sigma) = \bar{d}(\sigma)$ for every set σ of atoms of X where $\bar{d}$ is the usual simplicial differential with all summands $\pm\tau$ such that $\bigvee(\tau) = \bigvee(\sigma)$ omitted.

To check that ϕ is a ring homomorphism it suffices to check

$$f_{A\vee B}\bar{d}(\sigma \cdot \tau) = f_A\bar{d}(\sigma) \circ f_B\bar{d}(\tau) \tag{3.1}$$

for every sets σ and τ of atoms with $A = \bigvee(\sigma)$ and $B = \bigvee(\tau)$. We consider two cases.

(1) First suppose that $\sigma \cap \tau \neq \emptyset$ whence the left hand side of (3.1) is 0. To prove that so is the right hand side it suffices to consider a flag F constructed form a shuffle of two flags corresponding to some orders on sets $\sigma' = \sigma \setminus \{a\}$ and $\tau = \tau \setminus \{b\}$ respectively. Recall that by construction the largest element of F is either $\bigvee(\sigma') \vee B$ or $\bigvee(\tau') \vee A$ and in either of the cases this element is smaller than $A \vee B$. If $a \neq b$ then $\sigma' \cap \tau' \neq \emptyset$ and F has repetitions whence it is 0. Suppose $a = b$. Then $\bigvee(\sigma') \vee B = \bigvee(\sigma) \vee B = A \vee B = \bigvee(\tau') \vee A$ which is a contradiction.

(2) Now suppose that $\sigma \cap \tau = \emptyset$. First compare the sets of flags in the summands of the left and right hand sides of (3.1). Any (nonzero) flag F in the left hand side is constructed from an ordering of $(\sigma \cup \tau) \setminus \{a\}$. Suppose $a \in \sigma$ and denote by b the last element of τ in this ordering. Notice that $\pm\tau \setminus \{b\}$ is a summand in $\bar{d}(\tau)$ since otherwise F would have had repetitions. Then the induce orderings on $\sigma \setminus \{a\}$ and $\tau \setminus \{b\}$ produce flags F_1 and F_2 whose product is in the right hand side. The order used for F induces a shuffle that gives $\pm F$ as one of the summands in this product (the last term in this flag is $\bigvee(\sigma \setminus \{a\}) \vee \bigvee(\tau \setminus \{b\}) \vee B = \bigvee(\sigma \cup \tau \setminus \{a\})$. The case where $a \in \tau$ is considered similarly switching σ and τ. Thus we see that the set of flags on the left is included in the set of flags on the right. The opposite inclusion can be proved by inverting all the steps in the last proof.

Now we compare the coefficients of the flags. Let F be a flag as in the previous paragraph corresponding to an order on $(\sigma \setminus \{a\}) \cup \tau$ and b is the last element of τ. Let the coefficients of F in the left hand and right hand sides of (3.1) are $(-1)^\ell$ and $(-1)^r$ respectively. Then we have

$$\ell = \mathrm{sign}\epsilon(\sigma, \tau) + [a \in \sigma \cup \tau] + [(\sigma \setminus \{a\}) \cup \tau] \tag{3.2}$$

where the second term is the numerical position of a in $\sigma \cup \tau$ (in the initial ordering) and $[\rho]$ is the parity of the permutation induced by the ordering producing F on any subset ρ of $(\sigma \setminus \{a\}) \cup \tau$. Using similar notation we have

$$r = [a \in \sigma] + [b \in \tau] + [\sigma \setminus \{a\}] + [\tau \setminus \{b\}] + [\sigma \setminus \{a\}, \tau] + |\tau| \tag{3.3}$$

where $[\rho_1, \rho_2]$ is the parity of the shuffle induced by the ordering producing F of two disjoint subsets ρ_1 and ρ_2 of $(\sigma \setminus \{a\}) \cup (\tau \setminus \{b\})$ (from the starting position of ρ_2 after ρ_1). Recall that to obtain F we need first to augment $f_A(\sigma \setminus \{a\})$ by A at the end and $f_B(\tau \setminus \{b\})$ by B at the end and then apply the needed shuffle. It is easy to see (cf. the first part of the proof) that the shuffle should have A at the end of the set. Thus the last summand in (3.3) comes from moving A over τ. The augmentation by B amounts just to the substitution of b by B in τ.

Now we need the following straightforward equalities (or rather congruences modulo 2):

(a) $\text{sign}\epsilon(\sigma,\tau) = \text{sign}\epsilon(\sigma \setminus \{a\},\tau) + |\tau_{<a}|$
where the new symbols are self-explanatory (e.g., $\tau_{<a} = \{c \in \tau | c < a\}$);
(b) $[a \in \sigma \cup \tau] = [a \in \sigma] + |\tau_{<a}|$;
(c) $[\sigma \setminus \{a\}] + [\tau \setminus \{b\}] + [\sigma \setminus \{a\},\tau] = \text{sign}\epsilon(\sigma \setminus \{a\},\tau) + |\tau_{>b}| + [(\sigma \setminus \{a\}) \cup \tau]$;
(d) $[b \in \tau] + |\tau_{>b}| = |\tau|$.

Substituting (a) - (d) in (3.2) and (3.3) we obtain the needed equality of the coefficients of F in the left and right hand sides of (3.1). The proof of (3.1) for the case where $a \in \tau$ is similar. This completes the proof of the theorem. $\square$

I make the following conjecture.

Conjecture 3.3. *The ring $H^*(C(X),\mathbb{Z})$ is isomorphic to the cohomology ring of the differential graded ring with the underlying complex $\tilde{D}$ defined over $\mathbb{Z}$ and multiplication from the end of section 2. In particular this ring is defined by the lattice $L(X)$ labeled by the dimensions of its elements.*

This conjecture has been recently proved by Eva Feichtner in [7] for arrangements with geometric lattices and by Mark de Longueville in [6] for arrangements of coordinate subspaces.

In the rest of the section we consider a particular class of arrangements for which one can give a more explicit presentation of $H^*(C(X),\mathbb{Q})$. The complex $D = D(L)$ will be always considered over $\mathbb{Q}$.

Notice that any set $\sigma \subset \mathcal{A}$, such that

$$\bigvee(\sigma \setminus \{a\}) < \bigvee(\sigma) \tag{$*$}$$

for every $a \in \sigma$, is a cycle in D. If L is a geometric lattice then (*) defines independent sets of atoms. We will call these sets *independent* in general case also. For an independent set σ of atoms denote by ζ_σ its homology class in $H_*(D)$. Call a lattice L *homologically geometric* if classes ζ_σ generate the group $H_*(D)$. It is well known that geometric lattices are homologically geometric. In particular if X is an arrangement of hyperplanes then $L(X)$ is homologically geometric (cf. [12]). Another class of arrangements X whose lattices $L(X)$ are homologically geometric is formed by k-equal arrangements (see [12]).

Here is a couple of simple remarks about independent sets.

Lemma 3.4. *Let $\sigma \subset \mathcal{A}(L)$. Then the following properties are equivalent:*
(i) σ is independent;
(ii) every subset of σ is independent;
(iii) for every two distinct subsets τ_1 and τ_2 of σ we have $\bigvee(\tau_1) \neq \bigvee(\tau_2)$.

Proof. It is clear that (iii) implies (ii) and (ii) implies (i). Thus it suffices to prove that (i) implies (iii). Suppose σ is independent and there are $\tau_1, \tau_2 \subset \sigma$ such that

$\tau_1 \neq \tau_2$ but $\bigvee(\tau_1) = \bigvee(\tau_2)$. Let $A \in \tau_2 \setminus \tau_1$ and put $\tau_3 = \tau_1 \cup \{A\}$. Then $\bigvee(\tau_1) \leq \bigvee(\tau_3) \leq \bigvee(\tau_1) \vee \bigvee(\tau_2) = \bigvee(\tau_1)$ whence

$$\bigvee(\sigma \setminus \{A\}) = \bigvee(\tau_1) \vee \bigvee(\sigma \setminus \tau_3) = \bigvee(\tau_3) \vee \bigvee(\sigma \setminus \tau_3) = \bigvee(\sigma)$$

that contradicts (i). □

Lemma 3.5. *Let X be an arbitrary subset arrangement and σ and τ independent subsets of atoms of $L = L(X)$ with $\bigvee(\sigma) = A$ and $\bigvee(\tau) = B$. If $A \cap B = 0$ then $\sigma \cup \tau$ is independent.*

Proof. Put $\rho = \sigma \cup \tau$ and fix $C \in \rho$. We can assume that $C \in \sigma$ and put $A' = \bigvee(\sigma \setminus \{C\})$. Since σ is independent we have $A' < A$, in particular $\dim A' < \dim A$. Then using the condition on A and B we have

$$\dim \bigvee(\rho) = \dim(A + B) = \dim A + \dim B > \dim A' + \dim B$$

$$\geq \dim(A' + B) \geq \dim \bigvee(\rho \setminus \{C\})$$

whence $\bigvee(\rho) > \bigvee(\rho \setminus \{C\})$. This implies that ρ is independent. □

The classes ζ_σ for independent sets σ of atoms are not in general linearly independent. For instance if $\omega \in D$ is such that $d\omega$ can be written as a linear combination of independent sets then the respective linear combination r_ω of the homology classes is a linear relation. If such an element ω is besides homogeneous with respect to both (whence all three) degrees considered on D we call it a *relater*.

If the lattice $L(X)$ is homologically geometric then Theorem 3.2 and Lemma 3.5 give immediately a more explicit description of $H^*(C(X), \mathbb{Q})$.

Proposition 3.6. *(i) The space $H^*(\tilde{D})$ is generated by $\zeta_\sigma \in H^{2\dim U - |\sigma|}(D)$ where σ runs through all the independent sets of atoms of $L(X)$ and $U = \bigvee(\sigma)$. The linear relations among ζ_σ are generated by r_ω for the relaters ω.*

(ii) The other (non-linear) generating relations are $\zeta_\sigma\zeta_\tau = 0$ if $\dim \bigvee(\sigma \cup \tau) \neq \dim \bigvee(\sigma) + \dim \bigvee(\tau)$ and otherwise $\zeta_\sigma\zeta_\tau = (-1)^{\mathrm{sign}\,\epsilon(\sigma,\tau)}\zeta_{\sigma\cup\tau}$ where $\epsilon(\sigma,\tau)$ is as above the shuffle of $\sigma \cup \tau$ putting all the elements of τ after elements of σ.

Particular cases of that proposition are contained in Proposition 7.2 and Theorem 8.7 of [12].

4. Problems

1. **Naturality.** Theorem 3.2 does not solve the problem about the naturality of the isomorphism from its statement. This isomorphism becomes canonical after tensor multiplication by $\mathbb{C}$. In general situation considered by Morgan in [9] it is not natural over $\mathbb{Q}$. Whether it is natural in the situation of this paper is unknown to me. The most natural way to prove the naturality would be by showing that the mixed Hodge structure on $H^*(C(X), \mathbb{Q})$ splits into direct sum of pure structures of (p, p)-types.

2. **Integer coefficients.** The problem is to prove (or disprove) Conjecture 3.3 in all generality. For the case where the local homology of $L(X)$ is torsion-free the Conjecture would follow if one could prove that the direct sum decomposition of $H^*(C(X),\mathbb{Z})$ from [8] is canonical (i.e., does not depend on the choice of a Morse function).

3. **Homologically geometric lattices.** It would be interesting to discover other classes of arrangements with homologically geometric lattices. A natural candidate would be the class of arrangements of types $B_{n,k,h}$ from [2].

4. **Shellable lattices**. Shellable posets (EL-shellable, CL-shellable) form an extensively studied class of posets with explicitly defined generators of homology groups (see [3]). It would be interesting to obtain a presentation of $H^*(C(X))$ when $L(X)$ is shellable.

References

[1] K. Baclawski, *Whitney numbers of geometric lattices*, Adv. in Math. **16** (1975), 125–138.

[2] A. Björner and B. Sagan, *Subspace arrangements of type B_n and D_n*, J. Algebraic Combinatorics **5** (1996), 291–314.

[3] A. Björner and M. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), 323–341.

[4] A. Björner and J. Walker, *A homotopy complementation formula for partially ordered sets*, European J. Combin. **4** (1983), 11–19.

[5] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Mathematica, New Series **1**(1995), 459–494.

[6] M. De Longueville, *The ring structure on the cohomology of coordinate subspace arrangements*, preprint, 1998.

[7] E. M. Feichtner, *Cohomology algebras of subspace arrangements and of classical configuration spaces*, Cuvillier Verlag Göttingen, 1997 (Doctors Dissertation at TU, Berlin).

[8] M. Goresky and R. Mac Pherson, *Stratified Morse Theory, Part III*, Springer-Verlag, 1988.

[9] J. Morgan, *The algebraic topology of smooth algebraic varieties*, Int. Hautes Études Sci. Publ. Math. **48** (1978), 137–204.

[10] D. Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), 101–128.

[11] V. A. Vassiliev, *Complexes of connected graphs*, in The Gelfand's Mathematical Seminars, 1990–1992, Birkhäuser, Basel, 1993, 223–235.

[12] S. Yuzvinsky, *Small rational model of subspace complement*, preprint math. CO/9806143.

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