Projective representations of symmetric groups via Sergeev duality

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1 Introduction

In this article, we determine the irreducible projective representations of the symmetric group S_d and the alternating group A_d over an algebraically closed field of characteristic $p \neq 2$. These matters are well understood in the case $p = 0$, thanks to the fundamental work of Schur [24] in 1911, as well as the much more recent work of Nazarov [19, 20], Sergeev [25, 26] and others. So the focus here is primarily on the case of positive characteristic, where surprisingly little is known as yet. In particular, we obtain a natural combinatorial labelling of the irreducibles in terms of a certain set $\mathscr{RP}_p(d)$ of *restricted p-strict partitions of d*. Such partitions arose recently in work of Kashiwara, Miwa, Peterson and Yung [11] and Leclerc and Thibon [14] on the q-deformed Fock space of the affine Kac-Moody algebra of type $A^{(2)}_{p-1}$. Leclerc and Thibon proposed that $\mathscr{RP}_p(d)$ should label the irreducible projective representations in some natural way, and we show here how this can be done. Note that for $p = 3, 5$, the labelling problem was solved in [1, 3], while if $p = 2$ all projective representations of S_d and A_d are linear so do not need to be considered further here.

To be more precise, recall that λ is a partition of d if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a non-increasing sequence of non-negative integers summing to d. Call λ *p-strict* if in addition

$$\lambda_i = \lambda_{i+1} \quad \Rightarrow \quad p|\lambda_i \quad \text{for each } i = 1, 2, \dots.$$

Let $\mathscr{P}_p(d)$ denote the set of all p-strict partitions of d. Thus, the 0-strict partitions are just the partitions with no repeated non-zero parts, while a p-strict partition for $p > 0$ can only have repeated parts if they are divisible by p. Call $\lambda \in \mathscr{P}_p(d)$ a *restricted p-strict partition* if either $p = 0$, or $p > 0$ and

$$\begin{cases} \lambda_i - \lambda_{i+1} \leq p & \text{if } p \nmid \lambda_i, \\ \lambda_i - \lambda_{i+1} < p & \text{if } p \mid \lambda_i \end{cases}$$

for each $i = 1, 2, \dots$. Let $\mathscr{RP}_p(d) \subseteq \mathscr{P}_p(d)$ denote the restricted p-strict partitions of d. Also, define $h_{p'}(\lambda)$ to be the number of parts of λ not divisible by p. Then, our construction leads to a labelling of the irreducible projective representations of S_d over an algebraically closed field of characteristic $p \neq 2$ by pairs (λ, ε) where $\lambda \in \mathscr{RP}_p(d)$ and $\varepsilon = 0$ if $d - h_{p'}(\lambda)$

*Authors partially supported by the NSF (grant nos DMS-9801442 and DMS-9900134).

is even or ± 1 if $d - h_{p'}(\lambda)$ is odd. For A_d, the labelling is by pairs (λ, ε) where $\lambda \in \mathscr{RP}_p(d)$ and $\varepsilon = \pm 1$ if $d - h_{p'}(\lambda)$ even or 0 if $d - h_{p'}(\lambda)$ is odd.

The construction is based closely on the ideas of Sergeev and Nazarov in the characteristic 0 theory. In particular, the key step is to determine the irreducible "polynomial" representations of the supergroup $Q(n)$ in characteristic p. These turn out to be labelled naturally according to highest weight theory by *all* p-strict partitions with at most n non-zero parts. ¿From this, we use Sergeev's superalgebra analogue [25] of Schur-Weyl duality to determine the irreducible representations of a certain twisted version of the group algebra of the hyperoctahedral group. Finally, we pass from there to the symmetric group using methods of Nazarov [20] and Sergeev [26].

Acknowledgements. We would like to thank S. Ariki for bringing the work of Sergeev [25] to our attention, and S. Donkin for making [6, 18] available to us.

2 Preliminaries on superalgebras

In this section, we record a number of standard results about the representation theory of finite dimensional (associative) superalgebras. As useful general references, but sometimes with different conventions to us, we cite [17, ch.3], [15] and [10].

We will always work relative to a fixed algebraically closed field $\Bbbk$ of characteristic $p \neq 2$. By a *vector superspace* we mean a $\mathbb{Z}_2$-graded $\Bbbk$-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Given a homogeneous vector $0 \neq v \in V$, we denote its degree by $\partial(v) \in \mathbb{Z}_2$. A *subsuperspace* U of V means a subspace U of V such that $U = (U \cap V_{\bar{0}}) \oplus (U \cap V_{\bar{1}})$. Define the linear map $\delta_V : V \to V$ on homogeneous vectors by $\delta_V(v) = (-1)^{\partial(v)}v$. Then obviously, a subspace $U \subset V$ is a subsuperspace if and only if U is stable under δ_V.

Given vector superspaces V and W, we view the direct sum $V \oplus W$ and the tensor product $V \otimes W$ as a vector superspaces with $(V \oplus W)_i = V_i \oplus W_i$, and $(V \otimes W)_{\bar{0}} = V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}$, $(V \otimes W)_{\bar{1}} = V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}}$. Also, we make the vector space $\mathrm{Hom}_{\Bbbk}(V, W)$ of all linear maps from V to W into a superspace by declaring that $\mathrm{Hom}_{\Bbbk}(V, W)_i$ consists of the *homogeneous maps of degree* i for each $i \in \mathbb{Z}_2$, that is, the maps $\theta : V \to W$ with $\theta(V_j) \subseteq W_{i+j}$ for $j \in \mathbb{Z}_2$. Elements of $\mathrm{Hom}_{\Bbbk}(V, W)_{\bar{0}}$ will be referred to as *even* linear maps. The dual superspace V^* is $\mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, where we view $\Bbbk$ as a vector superspace concentrated in degree $\bar{0}$.

A *superalgebra* is a vector superspace A with the additional structure of an associative, unital $\Bbbk$-algebra such that $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. A *superalgebra homomorphism* $\theta : A \to B$ is an even linear map that is an algebra homomorphism in the usual sense; its kernel is a *superideal*, that is, an ordinary two-sided ideal that is also a subsuperspace. Most importantly, given two superalgebras A and B, we view the tensor product $A \otimes B$ as a superalgebra with the induced grading and multiplication defined by $(a \otimes b)(a' \otimes b') = (-1)^{\partial(b)\partial(a')}(aa') \otimes (bb')$ for homogeneous elements $a, a' \in A$, $b, b' \in B$. We note that $A \otimes B \cong B \otimes A$, an isomorphism being given by the *supertwist map* $T_{A,B} : A \otimes B \to B \otimes A, a \otimes b \mapsto (-1)^{\partial(a)\partial(b)} b \otimes a$ for homogeneous $a \in A, b \in B$.

2.1. **Example.** Let V be a vector superspace with $\dim V_{\bar{0}} = m, \dim V_{\bar{1}} = n$. The *tensor superalgebra* is the tensor algebra $T(V)$ regarded as a superalgebra with the induced grading.

As a quotient of $T(V)$, we have the *symmetric superalgebra*, namely,

$$S(V) = T(V)/\langle v \otimes w - (-1)^{\partial(v)\partial(w)} w \otimes v \mid \text{for all homogeneous vectors } v, w \in V\rangle.$$

If we have in mind fixed bases $v_1, \dots, v_m$ for $V_{\bar{0}}$ and $\bar{v}_1, \dots, \bar{v}_n$ for $V_{\bar{1}}$, we denote the superalgebras $T(V)$ and $S(V)$ instead by $T(m|n)$ and $S(m|n)$. These are the free superalgebra and the free commutative superalgebra on $m|n$ generators, respectively. Set $S(m) := S(m|0)$, just the usual polynomial algebra on m generators concentrated in degree $\bar{0}$, and $\bigwedge(n) := S(0|n)$, just the usual exterior algebra but with generators assigned the degree $\bar{1}$. The superalgebra $\bigwedge(n)$ is called the *Grassmann superalgebra*. We have that

$$\begin{aligned} S(m) &\cong S(1) \otimes \cdots \otimes S(1) \quad (m \text{ times}), \\ \textstyle\bigwedge(n) &\cong \textstyle\bigwedge(1) \otimes \cdots \otimes \textstyle\bigwedge(1) \quad (n \text{ times}), \\ S(m|n) &\cong S(m) \otimes \textstyle\bigwedge(n). \end{aligned}$$

2.2. **Example.** Another basic example that we will meet is the *Clifford superalgebra*, namely, the superalgebra $C(n)$ on generators $c_1, \dots, c_n$ all of degree $\bar{1}$, subject to the relations $c_i^2 = 1$ for $i = 1, \dots, n$ and $c_i c_j = -c_j c_i$ for all $i \neq j$. If, slightly more generally, one has in mind *non-zero* scalars $\lambda_1, \dots, \lambda_n \in \Bbbk^\times$, the superalgebra with generators $b_1, \dots, b_n$ subject to the relations $b_i^2 = \lambda_i, b_i b_j = -b_j b_i$ is isomorphic to $C(n)$, an obvious isomorphism sending $b_i \mapsto \sqrt{\lambda_i} c_i$. The crucial point is that $C(n_1 + n_2) \cong C(n_1) \otimes C(n_2)$. Indeed, the generators $c_1 \otimes 1, \dots, c_{n_1} \otimes 1, 1 \otimes c_1, \dots, 1 \otimes c_{n_2}$ of $C(n_1) \otimes C(n_2)$ satisfy the same relations as the generators $c_1, \dots, c_{n_1}, c_{n_1+1}, \dots, c_{n_1+n_2}$ of $C(n_1 + n_2)$. It follows at once that

$$C(n) \cong C(1) \otimes \cdots \otimes C(1) \quad (n \text{ times}).$$

Let A be a superalgebra. A *left A-supermodule* is a vector superspace M which is a left A-module in the usual sense, such that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. There is of course an analogous notion of right supermodule, which we omit. A homomorphism $f : M \to N$ between two left A-supermodules means a (not necessarily homogeneous) linear map such that $a(mf) = (am)f$ for all $a \in A$ and $m \in M$. Observe we write homomorphisms between *left* A-supermodules on the *right* (and vice versa). We have now defined the category $\mathbf{mod}(A)$ of all left A-supermodules. It is a *superadditive category* in the sense of [17, ch.3,§2.7], i.e. an additive category such that each $\mathrm{Hom}_A(M, N)$ is $\mathbb{Z}_2$-graded in a way that is compatible with addition and composition of morphisms. We also have the (left) *parity change functor*

$$\Pi : \mathbf{mod}(A) \to \mathbf{mod}(A)$$

(see [17, ch.3,§1.5]). This is defined on an object M so that ΠM is the same underlying vector space but with the opposite grading, and the new left A-action is defined by $a \cdot m = (-1)^{\partial(a)} am$ for homogeneous $a \in A, m \in M$. On a morphism f, Πf is the same underlying linear map as f.

A subsupermodule of an A-supermodule means an A-submodule in the usual sense that is a subsuperspace. An A-supermodule M is *irreducible* if it is non-zero and has no non-zero proper subsupermodules. Then M is either irreducible when viewed just as an ordinary A-module, in which case we say that M is *absolutely irreducible*, or else M is reducible as an A-module, in which case we call M *self-associate*.

2.3. **Lemma.** *If M is a finite dimensional self-associate irreducible A-supermodule, then there exist bases $v_1, \dots, v_n$ for $M_{\bar{0}}$ and $\bar{v}_1, \dots, \bar{v}_n$ for $M_{\bar{1}}$ such that*

$$M = \operatorname{span}\{v_1 + \bar{v}_1, \dots, v_n + \bar{v}_n\} \oplus \operatorname{span}\{v_1 - \bar{v}_1, \dots, v_n - \bar{v}_n\}$$

as a direct sum of two non-isomorphic irreducible A-submodules. Moreover, the linear map $J_M : M \to M$ defined by $v_i \mapsto \bar{v}_i, \bar{v}_i \mapsto v_i$ is an A-homomorphism.

Proof. We can find an irreducible A-submodule N of M that is not a subsupermodule, i.e. is not δ_M-stable. It is elementary to check that $\delta_M(N)$ is also an irreducible A-submodule of M. Hence, $N \oplus \delta_M(N)$ is an A-submodule of M, even a subsupermodule since it is now δ_M-stable. Let $u_1, \dots, u_n$ be a basis for N. Then, $\delta_M(u_1), \dots, \delta_M(u_n)$ is a basis for $\delta_M(N)$, so $u_1 + \delta_M(u_1), \dots, u_n + \delta_M(u_n)$ is the required basis for $M_{\bar{0}}$ and $u_1 - \delta_M(u_1), \dots, u_n - \delta_M(u_n)$ is the required basis for $M_{\bar{1}}$. $\square$

If M is an A-supermodule, $\operatorname{End}_A(M)$ denotes the superalgebra of all A-supermodule endomorphisms of M. We stress again that we write the action of elements of $\operatorname{End}_A(M)$ on M on the opposite side to the action of A. We have the following analogue of Schur's lemma, which is easily proved given Lemma 2.3:

2.4. **Lemma (Schur's lemma)**. *Let M be a finite dimensional irreducible A-supermodule. Then,*

$$\operatorname{End}_A(M) = \begin{cases} \operatorname{span}\{\mathrm{id}_M\} & \text{if } M \text{ is absolutely irreducible,} \\ \operatorname{span}\{\mathrm{id}_M, J_M\} & \text{if } M \text{ is self-associate,} \end{cases}$$

where J_M is as in Lemma 2.3.

We say that an A-supermodule M is completely reducible if it can be decomposed as a direct sum of irreducible subsupermodules. Call A a *simple superalgebra* if A has no non-trivial superideals, and a *semisimple superalgebra* if A is completely reducible viewed as a left A-supermodule. Equivalently, A is semisimple if *every* left A-supermodule is completely reducible. We have:

2.5. **Lemma (Wedderburn's theorem)**. *Let A be a finite dimensional superalgebra. The following are equivalent:*

(i) *A is simple;*

(ii) *A is semisimple with only one irreducible supermodule up to isomorphism;*

(iii) *there is a finite dimensional vector superspace V such that* either $A \cong \operatorname{End}_{\Bbbk}(V)$ or $A \cong \{\theta \in \operatorname{End}_{\Bbbk}(V) \mid \theta \circ J = J \circ \theta\}$ *for some involution $J \in \operatorname{End}_{\Bbbk}(V)_{\bar{1}}$.*

Moreover, if A is semisimple then it is isomorphic to a direct product of simple superalgebras.

Notice in view of Lemma 2.3 that if A is semisimple as a superalgebra, then it is semisimple as an algebra. The converse is also true, and is proved e.g. in [18, (1.4c)]; it can also be deduced directly by considering the effect of the map δ_A on the irreducible submodules of A viewed as a left A-module. Somewhat more generally, we have:

2.6. **Lemma.** *Let A be a finite dimensional superalgebra. Then, the Jacobson radical of A (viewed just as an ordinary algebra) can be characterized as the unique smallest superideal K of A such that A/K is a semisimple superalgebra.*

Proof. Let J be the Jacobson radical of A viewed as an ordinary algebra, and let K be any superideal of A that is minimal with respect to the property that A/K is a semisimple superalgebra. We know that A/K is semisimple as an ordinary algebra by Lemma 2.3, so $J \subseteq K$. Conversely, we observe that J is a superideal since J is invariant under the algebra automorphism δ_A of A. So, A/J is a superalgebra that is semisimple as an algebra. Hence, by [18, (1.4c)], it is a semisimple superalgebra, so $J = K$ by minimality of K. □

2.7. **Example.** The Jacobson radical of the Grassmann superalgebra $\bigwedge(n)$ coincides with the superideal generated by all degree $\bar{1}$ elements. The quotient superalgebra is isomorphic to $\Bbbk$. It follows that $\bigwedge(n)$ has a unique irreducible supermodule up to isomorphism, namely, $\Bbbk$ itself, with all elements of $\bigwedge(n)_{\bar{1}}$ acting as zero.

We point out another immediate consequence of Wedderburn's theorem and Lemma 2.6:

2.8. **Corollary.** *Let A be a finite dimensional superalgebra, and $\{V_1, \dots, V_n\}$ be a complete set of pairwise non-isomorphic irreducible A-supermodules such that $V_1, \dots, V_m$ are absolutely irreducible and $V_{m+1}, \dots, V_n$ are self-associate. For $i = m+1, \dots, n$, write $V_i = V_i^+ \oplus V_i^-$ as a direct sum of irreducible A-modules. Then,*

$$\{V_1, \dots, V_m, V_{m+1}^{\pm}, \dots, V_n^{\pm}\}$$

is a complete set of pairwise non-isomorphic irreducible A-modules.

Given left supermodules M and N over arbitrary superalgebras A and B respectively, the (outer) tensor product $M \otimes N$ is an $A \otimes B$-supermodule with action defined by $(a \otimes b)(m \otimes n) = (-1)^{\partial(b)\partial(m)} am \otimes bn$ for all homogeneous $a \in A, b \in B, m \in M, n \in N$. (Analogously, if M and N are right supermodules, the action of $A \otimes B$ on $M \otimes N$ is defined instead by $(m \otimes n)(a \otimes b) = (-1)^{\partial(a)\partial(n)} ma \otimes nb$ for all homogeneous $a \in A, b \in B, m \in M, n \in N$.) If $f : M \to M'$ (resp. $g : N \to N'$) is a homogeneous homomorphism of left A- (resp. B-) supermodules, then $f \otimes g : M \otimes N \to M' \otimes N'$ is the $A \otimes B$-supermodule homomorphism defined by $(m \otimes n)(f \otimes g) = (-1)^{\partial(n)\partial(f)} mf \otimes ng$. The following lemma gives the other basic facts about outer tensor products that we need (cf. [10, (2.10)]):

2.9. **Lemma.** *Suppose that A and B are finite dimensional superalgebras, and that M, N are irreducible supermodules over A, B respectively.*

(i) *If both M and N are absolutely irreducible, then $M \otimes N$ is an absolutely irreducible $A \otimes B$-supermodule.*

(ii) *If one of M or N is absolutely irreducible and the other is self-associate, then $M \otimes N$ is a self-associate irreducible $A \otimes B$-supermodule.*

(iii) *If both M and N are self-associate, then $M \otimes N$ decomposes as a direct sum of two isomorphic, absolutely irreducible $A \otimes B$-supermodules.*

Moreover, all irreducible $A \otimes B$-supermodules arise as constituents of $M \otimes N$ for some choice of M, N.

Combining Lemma 2.9 with Wedderburn's theorem, it follows in particular that if A and B are finite dimensional semisimple superalgebras then $A \otimes B$ is too.

2.10. **Example.** Consider the Clifford superalgebra $C(n)$ again. First, observe that $C(1)$ is just the simple superalgebra of 2×2 matrices of the form $\left\{ \left(\begin{array}{cc} a & b \\ b & a \end{array} \right) \middle| \, a, b \in \Bbbk \right\}$, the generator c_1 of $C(1)$ corresponding to the matrix $\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. So $C(1)$ has precisely one irreducible supermodule $U(1)$ which is self-associate of dimension 2, as in the second case of Lemma 2.5(iii). Hence, applying Lemma 2.9, $C(2) = C(1) \otimes C(1)$ has one irreducible supermodule $U(2)$, namely the unique irreducible appearing with multiplicity two in the $C(2)$-supermodule $U(1) \otimes U(1)$, and $U(2)$ is absolutely irreducible of dimension 2. Explicitly, $U(2)$ can be described as the supermodule on basis $u, \bar{u}$ with action defined by $c_1 u = \bar{u}, c_1 \bar{u} = u, c_2 u = \sqrt{-1}\bar{u}, c_2 \bar{u} = -\sqrt{-1} u$. Finally, for $n > 2$, $C(n) = C(n-2) \otimes C(2)$, so by Lemma 2.5(i) and (ii), it has just one irreducible supermodule $U(n)$, defined inductively by $U(n) = U(n-2) \otimes U(2)$. This is absolutely irreducible if and only if $U(n-2)$ is absolutely irreducible, which is if and only if n is even. Observe that we have just shown that $C(n)$ is a semisimple superalgebra with a unique irreducible supermodule. So by Lemma 2.5, $C(n)$ is in fact a simple superalgebra, indeed, up to isomorphism, it must be the unique simple superalgebra of dimension 2^n. Its unique irreducible supermodule $U(n)$ has dimension $2^{\lfloor (n+1)/2 \rfloor}$.

Following [25, §1.4], a $\mathbb{Z}_2$-*graded group* is a pair (G, ∂) where G is a finite group and $\partial : G \to \mathbb{Z}_2$ is a group homomorphism. If (G, ∂) is a $\mathbb{Z}_2$-graded group, we can regard the group algebra $\Bbbk G$ as a superalgebra, the degree of $g \in G$ being $\partial(g)$. We are interested next in counting the number of irreducible $\Bbbk G$-supermodules in terms of conjugacy classes. Define $n_{p'}(G, \bar{0})$ to be the number of G-conjugacy classes of p'-elements (= elements of order coprime to p) of degree $\bar{0}$ and $n_{p'}(G, \bar{1})$ to be the number of G-conjugacy classes of p'-elements of degree $\bar{1}$.

2.11. **Lemma.** *Let (G, ∂) be a $\mathbb{Z}_2$-graded group. Then, there are $n_{p'}(G, \bar{0})$ pairwise non-isomorphic irreducible $\Bbbk G$-supermodules. Of these, $n_{p'}(G, \bar{0}) - n_{p'}(G, \bar{1})$ are absolutely irreducible, and the remaining $n_{p'}(G, \bar{1})$ are self-associate.*

Proof. We follow the proof of the analogous classical result for ordinary group algebras, see [12, §13]. For an arbitrary superalgebra A, write $Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$ for its centre and $S(A) = \mathrm{span}\{ab - ba \mid a, b \in A\}$. These are both subsuperspaces of A. Let J denote the Jacobson radical of the group algebra $\Bbbk G$. By Lemma 2.6, J is a superideal and $A := \Bbbk G / J$ is the largest semisimple superalgebra quotient of $\Bbbk G$. So $\Bbbk G$ and A have the same number of irreducible supermodules. Combining Lemma 2.4 and Lemma 2.5, we deduce that the number of irreducible $\Bbbk G$-supermodules is equal to $\dim Z(A)_{\bar{0}}$ and the number of self-associate irreducible $\Bbbk G$-supermodules is equal to $\dim Z(A)_{\bar{1}}$. By [12, 13.3], $A = Z(A) \oplus S(A)$, so $\dim[Z(A)]_i = \dim[A/S(A)]_i$ for $i = \bar{0}, \bar{1}$. Finally, to count this dimension in either case, use formula (14) in the proof of [12, 13.8]; this tells us at once that $\dim[A/S(A)]_i = n_{p'}(G, i)$. $\square$

To conclude this preliminary section, we give a brief review of "Schur functors" arising from idempotents in this setting. Suppose that A is an arbitrary finite dimensional superalgebra and that $e \in A$ is a *homogeneous* idempotent, necessarily of degree $\bar{0}$. Then, the ring eAe is a superalgebra in its own right, its identity element being the idempotent e. We have the (exact) *Schur functor*

$$R_e : \mathbf{mod}(A) \to \mathbf{mod}(eAe)$$

given on objects by left multiplication by the idempotent e and by restriction on morphisms. Given an A-supermodule M, let $O_e(M)$ (resp. $O^e(M)$) denote the largest (resp. smallest) subsupermodule N of M such that N (resp. M/N) is annihilated by e. Finally, let $\mathbf{mod}_e(A)$ denote the full subcategory of $\mathbf{mod}(A)$ consisting of all A-supermodules M with $O_e(M) = 0$ and $O^e(M) = M$. The following basic result is proved as in the classical case, see [9, §2]:

2.12. **Lemma.** *The restriction of the functor R_e to $\mathbf{mod}_e(A)$ is an equivalence of categories between $\mathbf{mod}_e(A)$ and $\mathbf{mod}(eAe)$.*

Suppose that $\{L(\lambda) \mid \lambda \in \Lambda\}$ be a complete set of pairwise non-isomorphic irreducible A-supermodules, and set $\Lambda_e = \{\lambda \in \Lambda \mid R_e L(\lambda) \neq 0\}$. Then, as an immediate consequence of Lemma 2.12, we have:

2.13. **Corollary.** *The eAe-supermodules $\{R_e L(\lambda) \mid \lambda \in \Lambda_e\}$ give a complete set of pairwise non-isomorphic irreducible eAe-supermodules. Moreover, for $\lambda \in \Lambda_e$, $R_e L(\lambda)$ is absolutely irreducible if and only if $L(\lambda)$ is absolutely irreducible.*

3 The Sergeev superalgebra

Let S_d denote the symmetric group, acting naturally on the left on the set $\{1, \dots, d\}$. Denoting the basic transposition $(i\, i+1)$ by s_i, we recall that S_d is generated by $s_1, \dots, s_{d-1}$ subject to the well-known Coxeter relations.

Now let $\alpha : S_d \times S_d \to \Bbbk^\times$ be a 2-cocycle, where $\Bbbk$ is a fixed algebraically closed field. Then, there is a corresponding *twisted group algebra*, namely, the $\Bbbk$-algebra on basis $\{[w] \mid w \in S_d\}$ with multiplication satisfying $[x][y] = \alpha(x, y)[xy]$ for all $x, y \in S_d$. Studying the projective representations of S_d over $\Bbbk$ is equivalent to studying the representation theory of the twisted group algebras arising in this way, as α runs over representatives of all such 2-cocycles. The following lemma is quite standard, cf. [4] or [8, Kapitel 5, §25, Satz 12]:

3.1. **Lemma.** *The Schur multiplier $H^2(S_d, \Bbbk^\times)$ has exactly two elements if $\operatorname{char} \Bbbk \neq 2$ and $d \geq 4$, and is trivial otherwise.*

This explains in particular why all projective representations of S_d in characteristic 2 are linear, as remarked in the introduction. So now suppose for the remainder of the article that $\operatorname{char} \Bbbk \neq 2$. Then, Lemma 3.1 implies that S_d has two twisted group algebras over $\Bbbk$ up to isomorphism (providing $d \geq 4$). Of course, one of these is just the group algebra $\Bbbk S_d$

itself, and will not be considered further here. For the other, we may take the $\mathbb{k}$-algebra $S(d)$ on generators $t_1, \dots, t_{d-1}$ subject to the relations

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = -t_j t_i$$

for all $1 \le i \le d-1$ and all $1 \le j \le d-1$ with $|i-j| > 1$. In what follows, we will always view $S(d)$ as a *superalgebra*, defining the grading by declaring the generators $t_1, \dots, t_{d-1}$ to be of degree $\bar{1}$. We are interested in determining the irreducible $S(d)$-supermodules. Recall the definition of the set $\mathscr{RP}_p(d)$ of restricted p-strict partitions of d from the introduction.

3.2. **Lemma.** *The number of isomorphism classes of irreducible $S(d)$-supermodules is equal to $|\mathscr{RP}_p(d)|$.*

Proof. Define $\widehat{S}_d$ to be the double cover of S_d with generators $\zeta, \hat{s}_1, \dots, \hat{s}_{d-1}$ subject to the relations

$$\begin{aligned} \zeta^2 = \hat{s}_i^2 = 1, \qquad & \zeta \hat{s}_i = \hat{s}_i \zeta, \\ \hat{s}_i \hat{s}_{i+1} \hat{s}_i = \hat{s}_{i+1} \hat{s}_i \hat{s}_{i+1}, \qquad & \hat{s}_i \hat{s}_j = \zeta \hat{s}_j \hat{s}_i \end{aligned}$$

for all $1 \le i \le d-1$ and all $1 \le j \le d-1$ with $|i-j| > 1$ (see e.g. [27, p.100]). The map sending $\zeta \mapsto 1$, $\hat{s}_i \mapsto s_i$ determines a surjective homomorphism $\mathbb{k}\widehat{S}_d \to \mathbb{k}S_d$, while the map defined by $\zeta \mapsto -1$, $\hat{s}_i \mapsto t_i$ is a surjective homomorphism $\mathbb{k}\widehat{S}_d \to S(d)$.

Now, the elements $\zeta_+ = (1-\zeta)/2$ and $\zeta_- = (1+\zeta)/2$ are orthogonal central idempotents of $\mathbb{k}\widehat{S}_d$ summing to the identity, so

$$\mathbb{k}\widehat{S}_d = \zeta_+(\mathbb{k}\widehat{S}_d) \oplus \zeta_-(\mathbb{k}\widehat{S}_d)$$

as a direct sum of two-sided ideals. Obviously, $\zeta_+(\mathbb{k}\widehat{S}_d) \cong (\mathbb{k}\widehat{S}_d)/\langle \zeta - 1 \rangle \cong \mathbb{k}S_d$ and $\zeta_-(\mathbb{k}\widehat{S}_d) \cong S(d)$. Making S_d and $\widehat{S}_d$ into $\mathbb{Z}_2$-graded groups with degree function ∂ satisfying $\partial(\zeta) = \bar{0}$ and $\partial(\hat{s}_i) = \partial(s_i) = \bar{1}$, we deduce at once that the number of irreducible $\mathbb{k}\widehat{S}_d$-supermodules is equal to the number of irreducible $\mathbb{k}S_d$-supermodules plus the number of irreducible $S(d)$-supermodules. Hence, using Lemma 2.11, we deduce that the number of irreducible $S(d)$-supermodules is $n_{p'}(\widehat{S}_d, \bar{0}) - n_{p'}(S_d, \bar{0})$.

Finally, $n_{p'}(\widehat{S}_d, \bar{0}) - n_{p'}(S_d, \bar{0})$ can be calculated using the known labelling of the conjugacy classes of S_d and $\widehat{S}_d$, see e.g. [27, Theorem 2.1] or [24, p.172]. One deduces easily that the number of irreducible $S(d)$-supermodules is equal to the number of partitions λ of d with all non-zero parts of λ being odd and not divisible by p. In turn, to see that this number equals $|\mathscr{RP}_p(d)|$, we appeal to the partition identity

$$\sum_{d \ge 0} |\mathscr{RP}_p(d)| t^d = \prod_{i \text{ odd}, \, p \nmid i} \frac{1}{1-t^i},$$

from [14, (40)], which is a special case of [2, Theorem 2]. $\square$

Next, let $C(d)$ be the Clifford superalgebra on odd generators $c_1, \dots, c_d$ as in Example 2.2, so $c_i^2 = 1$ for each i. There is a unique right action of S_d on $C(d)$ by superalgebra automorphisms so that $c_i \cdot w = c_{w^{-1}i}$ for all $i = 1, \dots, d$ and $w \in S_d$. The *Sergeev superalgebra* is the vector superspace

$$W(d) = \Bbbk S_d \otimes C(d)$$

(here, $\Bbbk S_d$ is viewed as a superspace concentrated in degree $\bar{0}$) with multiplication defined on generators by the rule

$$(x \otimes c)(y \otimes d) = xy \otimes (c \cdot y)d$$

for $x, y \in S_d, c, d \in C(d)$. As observed by Sergeev in [26], a check of relations shows:

3.3. **Lemma.** *There is an injective superalgebra homomorphism $\omega : S(d) \to W(d)$ defined on generators by*

$$\omega(t_i) = \frac{1}{\sqrt{-2}} s_i \otimes (c_i - c_{i+1})$$

for each $i = 1, \dots, d-1$. Moreover, $\omega(t_i)(1 \otimes c_j) = -(1 \otimes c_j)\omega(t_i)$ for each $i = 1, \dots, d-1$ and $j = 1, \dots, d$.

Henceforth, we identify $S(d)$ with a subsuperalgebra of $W(d)$ via the embedding ω from the lemma, and also identify $C(d)$ with the subsuperalgebra $1 \otimes C(d)$ of $W(d)$. Then, Lemma 3.3 shows that multiplication defines a superalgebra isomorphism

$$C(d) \otimes S(d) \xrightarrow{\sim} W(d), \qquad c \otimes s \mapsto cs,$$

the tensor product of superalgebras on the left hand side being defined according to the usual rule of signs.

So we can define an exact functor

$$F : \mathbf{mod}(S(d)) \to \mathbf{mod}(W(d))$$

on an object M by $FM = U(d) \otimes M$, and on a morphism $f : M \to M'$ by $Ff = \mathrm{id}_{U(d)} \otimes f$. Thus, the action of a homogeneous $s \in S(d) \subset W(d)$ on $m \otimes u \in U(d) \otimes M$ is by $s(u \otimes m) = (-1)^{\partial(s)\partial(u)} u \otimes (sm)$, the action of $c \in C(d) \subset W(d)$ is by $c(u \otimes m) = (cu) \otimes m$, and $(u \otimes m)(\mathrm{id}_{U(d)} \otimes f) = u \otimes (mf)$. We also have an exact functor

$$G : \mathbf{mod}(W(d)) \to \mathbf{mod}(S(d)).$$

This is defined on an object N by $GN = \mathrm{Hom}_{C(d)}(U(d), N)$, the action of a homogeneous $s \in S(d)$ on $f \in \mathrm{Hom}_{C(d)}(U(d), N)$ being determined by $u(sf) = (-1)^{\partial(u)\partial(s)} s(uf)$ for all homogeneous $u \in U(d)$. On a morphism $g : N \to N'$, $Gg : \mathrm{Hom}_{C(d)}(U(d), N) \to \mathrm{Hom}_{C(d)}(U(d), N')$ is defined by $u(f(Gg)) = (uf)g$ for $u \in U(d)$ and $f \in \mathrm{Hom}_{C(d)}(U(d), N)$.

Recall the parity change functor Π defined in the previous section.

3.4. **Theorem.** *The functors F and G form an adjoint pair, that is, there is a natural (even) isomorphism*

$$\mathrm{Hom}_{W(d)}(FM, N) \cong \mathrm{Hom}_{S(d)}(M, GN)$$

for each $S(d)$-supermodule M and $W(d)$-supermodule N. Moreover:

(a) *if d is even, then $F \circ G \cong \mathrm{Id}$ and $G \circ F \cong \mathrm{Id}$;*

(b) *if d is odd, then $F \circ G \cong \mathrm{Id} \oplus \Pi$ and $G \circ F \cong \mathrm{Id} \oplus \Pi$.*

Proof. For adjointness, there are natural maps

$$\begin{aligned} \mathrm{Hom}_{W(d)}(U(d) \otimes M, N) &\to \mathrm{Hom}_{S(d)}(M, \mathrm{Hom}_{C(d)}(U(d), N)), \qquad & f \mapsto \hat{f}; \\ \mathrm{Hom}_{S(d)}(M, \mathrm{Hom}_{C(d)}(U(d), N)) &\to \mathrm{Hom}_{W(d)}(U(d) \otimes M, N), & g \mapsto \tilde{g}. \end{aligned}$$

Here, $\hat{f}$ is defined by $u(m\hat{f}) = (u \otimes m)f$ and $\tilde{g}$ is defined by $(u \otimes m)\tilde{g} = u(mg)$. Now check that $\tilde{\hat{f}} = f$ and $\hat{\tilde{g}} = g$.

Now we prove (b), the argument for (a) being similar (and considerably easier!). Let $E = \mathrm{End}_{C(d)}(U(d))$ for short, a vector superspace on basis $I = \mathrm{id}_{U(d)}, J = J_{U(d)}$ as in Lemma 2.4. On any category of left supermodules, the functor $\mathrm{Id} \oplus \Pi$ is naturally isomorphic to the tensor functor $E \otimes ?$, which sends a module M to $E \otimes_{\Bbbk} M$ and a morphism f to $\mathrm{id}_E \otimes f$ (written on the right). We will actually show that $G \circ F \cong E \otimes ?$ and that $F \circ G \cong E \otimes ?$.

We first show that $G \circ F \cong E \otimes ?$. Define a natural transformation $\eta : E \otimes ? \to G \circ F$ by defining the map

$$\eta_M : E \otimes M \to \mathrm{Hom}_{C(d)}(U(d), U(d) \otimes M)$$

for an $S(d)$-supermodule M by the formula $u\eta_M(f \otimes m) = uf \otimes m$ for each $u \in U(d)$, $m \in M, f \in E$. To see that η is actually a natural isomorphism, it suffices to consider the special case $M = \Bbbk$ when it is obvious.

Now we show that $F \circ G \cong E \otimes ?$. Define a natural transformation $\eta : F \circ G \to E \otimes ?$ by letting

$$\eta_N : U(d) \otimes \mathrm{Hom}_{C(d)}(U(d), N) \to E \otimes N$$

for each $W(d)$-supermodule N be the map

$$u \otimes f \mapsto I \otimes uf + (-1)^{\partial(u)} J \otimes uJf$$

for homogeneous $u \in U(d)$ and $f \in \mathrm{Hom}_{C(d)}(U(d), N)$. To see that η is actually a natural isomorphism, it suffices (since $C(d)$ is a simple superalgebra) to consider the special case $N = U(d)$. We can pick a homogeneous basis $u_1, \dots, u_n, \bar{u}_1, \dots, \bar{u}_n$ for $U(d)$ so that $u_i J = \bar{u}_i, \bar{u}_i J = u_i$ as in Lemma 2.3. Then, the map $\eta_{U(d)}$ maps $u_i \otimes I \mapsto I \otimes u_i + J \otimes \bar{u}_i$, $\bar{u}_i \otimes I \mapsto I \otimes \bar{u}_i - J \otimes u_i$, $u_i \otimes J \mapsto I \otimes \bar{u}_i + J \otimes u_i$ and $\bar{u}_i \otimes J \mapsto I \otimes u_i - J \otimes \bar{u}_i$. It is obvious from this that it is a bijection. □

3.5. **Corollary.** (a) *Suppose d is even. The functors F and G induce mutually inverse bijections between isomorphism classes of irreducible (resp. absolutely irreducible) $S(d)$-supermodules and irreducible (resp. absolutely irreducible) $W(d)$-supermodules.*

(b) *Suppose d is odd. The functor F induces a bijection between isomorphism classes of absolutely irreducible $S(d)$-supermodules and self-associate irreducible $W(d)$-supermodules. The functor G induces a bijection between isomorphism classes of absolutely irreducible $W(d)$-supermodules and self-associate irreducible $S(d)$-supermodules.*

Proof. (a) This is obvious since F and G are mutually inverse equivalences of categories.

(b) Let D be an irreducible $S(d)$-supermodule. By Lemma 2.9, FD is a self-associate irreducible $W(d)$-supermodule in case D is absolutely irreducible, and decomposes as a direct sum of two isomorphic absolutely irreducible $W(d)$-supermodules in case D is self-associate.

It is now straightforward to complete the proof of the corollary using the properties of F and G from Theorem 3.4. $\square$

For the remainder of the article, we in fact work with the Sergeev superalgebra $W(d)$ instead of with $S(d)$, this being justified by Theorem 3.4 and its corollary. To conclude the section, we develop notation for products of arbitrary elements in $W(d)$.

First, let W_d denote the *hyperoctahedral group*, that is, the semidirect product of S_d and $\mathbb{Z}_2^d$. To be more precise, denote elements of the Abelian group $\mathbb{Z}_2^d$ by d-tuples $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ with each $\varepsilon_i \in \mathbb{Z}_2$. There is a right action of S_d on $\mathbb{Z}_2^d$ given by $\varepsilon \cdot w = (\varepsilon_{w1}, \varepsilon_{w2}, \dots, \varepsilon_{wd})$ for $w \in S_d, \varepsilon \in \mathbb{Z}_2^d$. Then, elements of W_d are pairs (w, ε) with $w \in S_d, \varepsilon \in \mathbb{Z}_2^d$, the product of two such elements being defined by

$$(x, \varepsilon)(y, \delta) = (xy, \varepsilon \cdot y + \delta).$$

Henceforth, we will identify $w \in S_d$ (resp. $\varepsilon \in \mathbb{Z}_2^d$) with the element $(w, 0) \in W_d$ (resp. $(1, \varepsilon) \in W_d$). It will also be convenient to extend the action of S_d on $\mathbb{Z}_2^d$ to an action of all of W_d on $\mathbb{Z}_2^d$, so that $\varepsilon \cdot (w, \delta) = \varepsilon \cdot w + \delta$ for $\varepsilon \in \mathbb{Z}_2^d, (w, \delta) \in W_d$.

For $\varepsilon \in \mathbb{Z}_2^d$, let

$$c^\varepsilon = c_1^{\varepsilon_1} \dots c_d^{\varepsilon_d} \in C(d)$$

Then, the $\{c^\varepsilon \mid \varepsilon \in \mathbb{Z}_2^d\}$ form a basis for the Clifford superalgebra $C(d)$. The product of two such basis elements is given explicitly by the rule

$$c^\varepsilon c^\delta = \alpha(\varepsilon; \delta) c^{\varepsilon + \delta} \qquad \text{where} \qquad \alpha(\varepsilon; \delta) = \prod_{1 \leq s < t \leq d} (-1)^{\delta_s \varepsilon_t}$$

for $\varepsilon, \delta \in \mathbb{Z}_2^d$. It is worth remarking for later calculations that $\alpha(\varepsilon + \varepsilon'; \delta) = \alpha(\varepsilon; \delta)\alpha(\varepsilon'; \delta)$ and $\alpha(\varepsilon; \delta + \delta') = \alpha(\varepsilon; \delta)\alpha(\varepsilon; \delta')$.

We obtain a basis $\{w \otimes c^\varepsilon \mid w \in S_d, \varepsilon \in \mathbb{Z}_2^d\}$ for the Sergeev superalgebra $W(d) = \Bbbk S_d \otimes C(d)$. The right action of $w \in S_d$ on the basis element c^ε of $C(d)$ is given by the formula

$$c^\varepsilon \cdot w = \alpha(\varepsilon; w) c^{\varepsilon \cdot w} \qquad \text{where} \qquad \alpha(\varepsilon; w) = \prod_{\substack{1 \leq s < t \leq d \\ w^{-1}s > w^{-1}t}} (-1)^{\varepsilon_s \varepsilon_t}.$$

Hence, the product of two basis elements of $W(d)$ given by the formula

$$(x \otimes c^\varepsilon)(y \otimes c^\delta) = \alpha(x, \varepsilon; y, \delta) xy \otimes c^{\varepsilon \cdot y + \delta} \qquad \text{where} \qquad \alpha(x, \varepsilon; y, \delta) = \alpha(\varepsilon; y)\alpha(\varepsilon \cdot y; \delta).$$

It follows that the resulting function $\alpha : W_d \times W_d \to \{\pm 1\}$, $((x, \varepsilon), (y, \delta)) \mapsto \alpha(x, \varepsilon; y, \delta)$ is a 2-cocycle on W_d. So $W(d)$ is a twisted group algebra of the hyperoctahedral group W_d over $\Bbbk$. In particular, the twisted group algebra analogue of Maschke's theorem gives:

3.6. **Lemma.** *If $p = 0$ or $p > d$, then $W(d)$ is a semisimple (super)algebra.*

We finally record a technical property about the cocycle α just constructed.

3.7. **Lemma.** *For all $\varepsilon, \delta \in \mathbb{Z}_2^d$ and $g = (w, \gamma) \in W_d$,*

$$\alpha(\varepsilon + \delta; w) = \alpha(\varepsilon; g)\alpha(\delta; g)\alpha(\varepsilon + \delta; \delta)\alpha(\varepsilon \cdot g + \delta \cdot g; \delta \cdot g).$$

Proof. Expand both sides of the equation $(c^{\varepsilon+\delta}c^\delta)\cdot w = (c^{\varepsilon+\delta}\cdot w)(c^\delta\cdot w)$ to show that $\alpha(\varepsilon+\delta;w) = \alpha(\varepsilon;w)\alpha(\delta;w)\alpha(\varepsilon+\delta;\delta)\alpha(\varepsilon\cdot w+\delta\cdot w;\delta\cdot w)$. Now expand the definition of $\alpha(\varepsilon;g)\alpha(\delta;g)\alpha(\varepsilon\cdot g+\delta\cdot g;\delta\cdot g)$ to see that it equals

$$\begin{aligned}
&\alpha(\varepsilon;w)\alpha(\varepsilon\cdot w;\gamma)\alpha(\delta;w)\alpha(\delta\cdot w;\gamma)\alpha(\varepsilon\cdot w+\delta\cdot w;\delta\cdot w+\gamma)\\
&=\alpha(\varepsilon;w)\alpha(\delta;w)\alpha(\varepsilon\cdot w+\delta\cdot w;\delta\cdot w)\alpha(\varepsilon\cdot w;\gamma)\alpha(\delta\cdot w;\gamma)\alpha(\varepsilon\cdot w+\delta\cdot w;\gamma)\\
&=\alpha(\varepsilon;w)\alpha(\delta;w)\alpha(\varepsilon\cdot w+\delta\cdot w;\delta\cdot w),
\end{aligned}$$

and the result follows. $\square$

4 The Schur superalgebra

We introduce some further notation. Suppose that $0\neq i,j\in\mathbb{Z}$. Define $\partial_i=\bar{0}$ if $i>0$ or $\bar{1}$ if $i<0$; define $\partial_{i,j}=\partial_i+\partial_j\in\mathbb{Z}_2$. More generally, given d-tuples $\underline{i}=(i_1,\dots,i_d)$ and $\underline{j}=(j_1,\dots,j_d)$ of non-zero integers, let

$$\begin{aligned}
\partial_{\underline{i}} &= \partial_{i_1}+\cdots+\partial_{i_d}\in\mathbb{Z}_2, &\qquad \partial_{\underline{i},\underline{j}} &= \partial_{\underline{i}}+\partial_{\underline{j}}\in\mathbb{Z}_2,\\
\varepsilon_{\underline{i}} &= (\partial_{i_1},\partial_{i_2},\dots,\partial_{i_d})\in\mathbb{Z}_2^d, &\qquad \varepsilon_{\underline{i},\underline{j}} &= \varepsilon_{\underline{i}}+\varepsilon_{\underline{j}}\in\mathbb{Z}_2^d.
\end{aligned}$$

Let $\mathbb{Z}_2^d$ act on the left on $\{\pm1,\dots,\pm d\}$ so that for $\varepsilon=(\varepsilon_1,\dots,\varepsilon_d)\in\mathbb{Z}_2^d$ and $s=1,\dots,d$, $\varepsilon(\pm s)=(-1)^{\varepsilon_s}(\pm s)$. Extend the natural action of S_d on $\{1,\dots,d\}$ to an action on $\{\pm1,\dots,\pm d\}$ so that $w(-s)=-(ws)$ for $s=1,\dots,d$. These two actions combine to give a well-defined left action of the hyperoctahedral group W_d on the set $\{\pm1,\dots,\pm d\}$.

Now let $I(n,d)$ denote the set of all functions $\underline{i}:\{\pm1,\dots,\pm d\}\to\{\pm1,\dots,\pm n\}$ such that $\underline{i}(-s)=-\underline{i}(s)$ for $s=1,\dots,d$. We often denote the value $\underline{i}(s)$ of the function $\underline{i}\in I(n,d)$ at $s\in\{\pm1,\dots,\pm d\}$ by i_s. Then, the element $\underline{i}\in I(n,d)$ can be thought of simply as the d-tuple $(i_1,\dots,i_d)$: the original function $\underline{i}$ can be recovered uniquely from knowledge of this d-tuple since $\underline{i}(-s)=-\underline{i}(s)$. The group W_d acts on the right on $I(n,d)$ by composition of functions, so $(\underline{i}\cdot g)(s)=\underline{i}(gs)$ for $\underline{i}\in I(n,d), g\in W_d$ and $s\in\{\pm1,\dots,\pm d\}$. Write $\underline{i}\sim\underline{j}$ if $\underline{i},\underline{j}\in I(n,d)$ lie in the same W_d-orbit. Also let W_d act diagonally on the right on the set $I(n,d)\times I(n,d)$ of *double indexes*, and write $(\underline{i},\underline{j})\sim(\underline{k},\underline{l})$ if the double indexes $(\underline{i},\underline{j})$ and $(\underline{k},\underline{l})$ lie in the same orbit.

Let V denote the vector superspace with basis $v_{\pm1},\dots,v_{\pm n}$, where $\partial(v_i)=\partial_i$. Then, the tensor product $V^{\otimes d}$ is also a vector superspace with the induced grading. A basis is given by the monomials $v_{\underline{i}}=v_{i_1}\otimes\cdots\otimes v_{i_d}$ for all $\underline{i}\in I(n,d)$, and $\partial(v_{\underline{i}})=\partial_{\underline{i}}$. We make $V^{\otimes d}$ into a *right* $W(d)$-supermodule by setting

$$v_{\underline{i}}(w\otimes c^\delta)=\alpha(\varepsilon_{\underline{i}};w,\delta)v_{\underline{i}\cdot(w,\delta)}$$

for all $\underline{i}\in I(n,d),(w,\delta)\in W_d$. The fact that this is well-defined follows from the fact that α is a 2-cocycle. To be more explicit, the action of the generator s_i of $S_d\subset W(d)$ is as the linear map $\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes T_{V,V}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}$ where the supertwist map $T_{V,V}$ is in the ith position, and the generator c_j of $C(d)\subset W(d)$ acts on the right (with our usual convention regarding signs) as the linear map $\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes J_V\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}$ where the map $J_V: v_i\mapsto v_{-i}$ is in the jth tensor.

Now define the *Schur superalgebra* of type Q

$$\dot{Q}(n,d) := \mathrm{End}_{W(d)}(V^{\otimes d}).$$

So, $\dot{Q}(n,d)$ acts on $V^{\otimes d}$ on the *left*. Note $\dot{Q}(n,d)$ is naturally a subsuperalgebra of the Schur superalgebra $\dot{S}(n|n,d) = \mathrm{End}_{\Bbbk S_d}(V^{\otimes d})$ of type GL, which was studied in [18, 6]. We observe right away by Lemma 3.6 that:

4.1. **Lemma.** *If $p = 0$ or $p > d$, then $\dot{Q}(n,d)$ is a semisimple (super)algebra.*

The initial goal is to describe an explicit basis for $\dot{Q}(n,d)$.

4.2. **Lemma.** *For $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$, the following properties are equivalent:*
(i) $\partial_{i_s,j_s}\partial_{i_t,j_t} = \bar{0}$ *whenever $|i_s| = |i_t|$ and $|j_s| = |j_t|$ for some $1 \le s < t \le d$;*
(ii) $\alpha(\varepsilon_{\underline{i},\underline{j}}; w) = 1$ *for all $(w,\delta) \in \mathrm{Stab}_{W_d}(\underline{i},\underline{j})$.*

Proof. Using the fact that $\mathrm{Stab}_{S_d}(\underline{i},\underline{j})$ is generated by transpositions and that α is a 2-cocycle, property (ii) is equivalent to the condition that $\alpha(\varepsilon_{\underline{i},\underline{j}}; w) = 1$ for all $(w,\delta) \in \mathrm{Stab}_{W_d}(\underline{i},\underline{j})$ with w a transposition. This weaker statement is precisely the condition (i), by the definition of α. $\square$

Call the double index $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$ *strict* if it satisfies the properties in the lemma, and let $I^2(n,d)$ denote the set of all strict double indexes. Observe using Lemma 4.2(i) that $I^2(n,d)$ is W_d-stable. Given $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l}) \in I^2(n,d)$, choose $(w,\delta) \in W_d$ such that $(\underline{i},\underline{j}) \cdot (w,\delta) = (\underline{k},\underline{l})$ and define the sign $\sigma(\underline{i},\underline{j};\underline{k},\underline{l})$ to be $\alpha(\varepsilon_{\underline{i},\underline{j}}; w)$. In view of Lemma 4.2(ii), this definition of $\sigma(\underline{i},\underline{j};\underline{k},\underline{l})$ is independent of the choice of (w,δ).

Given $i,j \in \{\pm 1,\dots,\pm d\}$, let $\dot{e}_{i,j} \in \mathrm{End}_{\Bbbk}(V)$ denote the linear map with $\dot{e}_{i,j}v_k = \delta_{j,k}v_i$ for all k. Given $\underline{i},\underline{j} \in I(n,d)$, let

$$\dot{e}_{\underline{i},\underline{j}} = \dot{e}_{i_1,j_1} \otimes \dot{e}_{i_2,j_2} \otimes \cdots \otimes \dot{e}_{i_d,j_d} \in \mathrm{End}_{\Bbbk}(V)^{\otimes d}.$$

Now there is an isomorphism between the superalgebras $\mathrm{End}_{\Bbbk}(V)^{\otimes d}$ and $\mathrm{End}_{\Bbbk}(V^{\otimes d})$ under which our element $\dot{e}_{\underline{i},\underline{j}} \in \mathrm{End}_{\Bbbk}(V)^{\otimes d}$ corresponds to the linear map $V^{\otimes d} \to V^{\otimes d}$ with

$$\dot{e}_{\underline{i},\underline{j}}v_{\underline{k}} = \delta_{\underline{j},\underline{k}}\alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{j}})v_{\underline{i}}. \tag{4.3}$$

We will henceforth identify $\mathrm{End}_{\Bbbk}(V)^{\otimes d}$ and $\mathrm{End}_{\Bbbk}(V^{\otimes d})$ in this way. Given *strict* $(\underline{i},\underline{j}) \in I^2(n,d)$, define the linear map $\dot{\xi}_{\underline{i},\underline{j}} \in \mathrm{End}_{\Bbbk}(V^{\otimes d})$ by

$$\dot{\xi}_{\underline{i},\underline{j}} = \sum_{(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})} \sigma(\underline{i},\underline{j};\underline{k},\underline{l})\dot{e}_{\underline{k},\underline{l}}. \tag{4.4}$$

Obviously, if $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l}) \in I^2(n,d)$, then $\dot{\xi}_{\underline{i},\underline{j}} = \sigma(\underline{i},\underline{j};\underline{k},\underline{l})\dot{\xi}_{\underline{k},\underline{l}}$. Now choose some set $\Omega(n,d)$ of orbit representatives for the action of W_d on $I^2(n,d)$. Then:

4.5. **Theorem.** *The elements* $\{\dot\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in \Omega(n,d)\}$ *give a basis for* $\dot Q(n,d)$. *Moreover, given* $(\underline{i},\underline{j}),(\underline{k},\underline{l}) \in I^2(n,d)$,

$$\dot\xi_{\underline{i},\underline{j}}\dot\xi_{\underline{k},\underline{l}} = \sum_{(\underline{s},\underline{t})\in\Omega(n,d)} a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}}\dot\xi_{\underline{s},\underline{t}}$$

where

$$a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}} = \sum_{\substack{\underline{h}\in I(n,d) \text{ with} \\ (\underline{s},\underline{h})\sim(\underline{i},\underline{j}),(\underline{h},\underline{t})\sim(\underline{k},\underline{l})}} \sigma(\underline{i},\underline{j};\underline{s},\underline{h})\sigma(\underline{k},\underline{l};\underline{h},\underline{t})\alpha(\varepsilon_{\underline{s},\underline{h}};\varepsilon_{\underline{h},\underline{t}}).$$

Proof. Obviously, the given elements are linearly independent. To show that they span $\mathrm{End}_{W(d)}(V^{\otimes d})$, let

$$\theta = \sum_{\underline{i},\underline{j}\in I(n,d)} a_{\underline{i},\underline{j}}\dot e_{\underline{i},\underline{j}}$$

be an arbitrary element of $\mathrm{End}_{\Bbbk}(V^{\otimes d})$. Take $w \in S_d, \delta \in \mathbb{Z}_2^d$ and set $g = (w,\delta) \in W_d$. For $\underline{j} \in I(n,d)$, we have that $(\theta v_{\underline{j}})(w \otimes c^\delta) = \theta(v_{\underline{j}}(w \otimes c^\delta))$ if and only if

$$\sum_{\underline{i}\in I(n,d)} a_{\underline{i},\underline{j}}\alpha(\varepsilon_{\underline{i},\underline{j}};\varepsilon_{\underline{j}})\alpha(\varepsilon_{\underline{i}};g)v_{\underline{i}\cdot g} = \sum_{\underline{i}\in I(n,d)} a_{\underline{i}\cdot g,\underline{j}\cdot g}\alpha(\varepsilon_{\underline{i}\cdot g,\underline{j}\cdot g};\varepsilon_{\underline{j}\cdot g})\alpha(\varepsilon_{\underline{j}};g)v_{\underline{i}\cdot g}$$

Simplifying using Lemma 3.7, we see that $\theta \in \mathrm{End}_{W(d)}(V^{\otimes d})$ if and only if

$$a_{\underline{i}\cdot g,\underline{j}\cdot g} = \alpha(\varepsilon_{\underline{i},\underline{j}};w)a_{\underline{i},\underline{j}}$$

for all $\underline{i},\underline{j} \in I(n,d)$ and $g = (w,\delta) \in W_d$. So by Lemma 4.2(ii), we must have that $a_{\underline{i},\underline{j}} = 0$ unless $(\underline{i},\underline{j})$ is strict, and for strict $(\underline{h},\underline{k}) \sim (\underline{i},\underline{j})$, we have that $a_{\underline{h},\underline{k}} = \sigma(\underline{i},\underline{j};\underline{h},\underline{k})a_{\underline{i},\underline{j}}$. This shows that $\theta \in \dot Q(n,d)$ if and only if $\theta = \sum_{(\underline{i},\underline{j})\in\Omega(n,d)} a_{\underline{i},\underline{j}}\dot\xi_{\underline{i},\underline{j}}$, completing the proof of the first part of the theorem.

Now we show how to deduce the product rule. To calculate $a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}}$ in the product expansion, we need by (4.4) to determine the coefficient of $\dot e_{\underline{s},\underline{t}}$ in

$$\dot\xi_{\underline{i},\underline{j}}\dot\xi_{\underline{k},\underline{l}} = \sum_{(\underline{i}',\underline{j}')\sim(\underline{i},\underline{j})}\ \sum_{(\underline{k}',\underline{l}')\sim(\underline{k},\underline{l})} \sigma(\underline{i},\underline{j};\underline{i}',\underline{j}')\sigma(\underline{k},\underline{l};\underline{k}',\underline{l}')\dot e_{\underline{i}',\underline{j}'}\dot e_{\underline{k}',\underline{l}'}.$$

We have that $\dot e_{\underline{i}',\underline{j}'}\dot e_{\underline{k}',\underline{l}'} = \delta_{\underline{j}',\underline{k}'}\alpha(\varepsilon_{\underline{i}',\underline{j}'};\varepsilon_{\underline{k}',\underline{l}'})\dot e_{\underline{i}',\underline{l}'}$. Using this the $\dot e_{\underline{s},\underline{t}}$-coefficient of $\dot\xi_{\underline{i},\underline{j}}\dot\xi_{\underline{k},\underline{l}}$ is therefore precisely as in the theorem (with $\underline{h} = \underline{j}' = \underline{k}'$). $\square$

5 The coordinate ring

Now we proceed to give an entirely different construction of the Schur superalgebra in the spirit of Green's monograph [7]. We begin by reviewing some basic facts about cosuperalgebras and bisuperalgebras, following [18].

A *cosuperalgebra* is a vector superspace A with the additional structure of a $\Bbbk$-coalgebra, such that both the comultiplication $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to \Bbbk$ are even linear maps. Given two cosuperalgebras A and B, $A \otimes B$ is a cosuperalgebra with comultiplication $\mathrm{id}_A \otimes T_{A,B} \otimes \mathrm{id}_B \circ (\Delta_A \otimes \Delta_B)$. A cosuperalgebra homomorphism $\theta : A \to B$ means an even linear map that is a coalgebra homomorphism in the usual sense. Cosuperideals and subcosuperalgebras are also the obvious graded version of the usual notions.

Given a cosuperalgebra A, a right A-cosupermodule is a vector superspace M together with an even linear map $\eta : M \to M \otimes A$, called the *structure map* of M, which makes M into a right A-comodule in the usual sense. A homomorphism between two A-cosupermodules means an A-comodule homomorphism in the usual sense; note we write homomorphisms between right A-cosupermodules on the *left* (and vice versa). We let $\mathbf{comod}(A)$ denote the (superadditive) category of all right A-cosupermodules.

A *bisuperalgebra* is a vector superspace A that is both a superalgebra and a cosuperalgebra, such that the comultiplication $\Delta : A \to A \otimes A$ (recall how $A \otimes A$ is viewed as a superalgebra!) and counit $\epsilon : A \to \Bbbk$ are superalgebra homomorphisms. If A is a bisuperalgebra, we have a natural notion of (inner) tensor product of two right A-cosupermodules M and N, namely, the vector superspace $M \otimes N$ with structure map defined by the composition

$$M \otimes N \stackrel{\eta_M \otimes \eta_N}{\longrightarrow} M \otimes A \otimes N \otimes A \stackrel{\mathrm{id} \otimes T_{A,N} \otimes \mathrm{id}}{\longrightarrow} M \otimes N \otimes A \otimes A \stackrel{\mathrm{id} \otimes \mathrm{id} \otimes \mu}{\longrightarrow} M \otimes N \otimes A,$$

where $\eta_M : M \to M \otimes A$ and $\eta_N : N \to N \otimes A$ are the structure maps of M, N, respectively, and $\mu : A \otimes A \to A$ denotes the multiplication in A (one needs to know here that μ is a cosuperalgebra homomorphism, see e.g. [6, §2.2]).

Let A be a finite dimensional cosuperalgebra. We make the dual superspace A^* into a superalgebra by defining the product $f_1 f_2$ of homogeneous $f_1, f_2 \in A^*$ by $(f_1 f_2)(a) = (f_1 \bar{\otimes} f_2)\Delta(a)$, interpreting the right hand side according to the usual rule of signs. Given a right A-cosupermodule M with structure map $\eta : M \to M \otimes A$, we can view M as a left A^*-supermodule, with action defined by $fm = (\mathrm{id}_M \bar{\otimes} f)\eta(m)$ for $f \in A^*, m \in M$. Now suppose that $\theta : M \to N$ is a homogeneous morphism of right A-cosupermodules and define $\tilde{\theta} : M \to N$ by $m\tilde{\theta} := (-1)^{\partial(m)\partial(\theta)}\theta m$ for homogeneous $m \in M$. Then, viewing M and N as left A^*-supermodules as just explained, the map $\tilde{\theta}$ is a morphism of left A^*-supermodules. One obtains in this way an *isomorphism* between the categories $\mathbf{comod}(A)$ and $\mathbf{mod}(A^*)$.

Finally in this review of definitions, we mention a standard general result about direct sums of cosuperalgebras. Suppose A is a (possibly infinite dimensional) cosuperalgebra and that $A = \bigoplus_{i \in I} A_i$ as a direct sum of subcosuperalgebras. Then, as in [7, p.20] we have:

5.1. **Lemma.** *With the preceeding notation, let M be a right A-cosupermodule with structure map $\eta : M \to M \otimes A$. Then, $M = \bigoplus_{i \in I} M_i$ where M_i is the unique maximal subcosupermodule of M with $\eta(M_i) \subseteq M_i \otimes A_i$.*

As a corollary, one obtains that the category of right A-cosupermodules is equivalent to the product of the categories of right A_i-cosupermodules for all $i \in I$.

Now we begin the alternative construction of the Schur superalgebra. Start with the free superalgebra $F(n)$ on non-commuting generators $\{f_{i,j} \mid i, j = \pm 1, \dots, \pm n\}$, where $\partial(f_{i,j}) = \partial_{i,j}$. Then, $F(n)$ is naturally $\mathbb{Z}$-graded by degree as

$$F(n) = \bigoplus_{d \geq 0} F(n, d).$$

Given a double index $(\underline{i}, \underline{j}) \in I(n,d) \times I(n,d)$, define $f_{\underline{i},\underline{j}} = f_{i_1,j_1} f_{i_2,j_2} \dots f_{i_d,j_d}$. The elements $\{f_{\underline{i},\underline{j}} \mid (\underline{i}, \underline{j}) \in I(n,d) \times I(n,d)\}$ form a basis for $F(n,d)$. One checks that the unique superalgebra maps $\epsilon : F(n) \to \Bbbk$ and $\Delta : F(n) \to F(n) \otimes F(n)$ defined on generators by

$$\begin{aligned} \epsilon(f_{i,j}) &= \delta_{i,j}, \\ \Delta(f_{i,k}) &= \sum_{j \in \{\pm 1, \dots, \pm n\}} (-1)^{\partial_{i,j} \partial_{j,k}} f_{i,j} \otimes f_{j,k} \end{aligned}$$

make $F(n)$ into a bisuperalgebra. We point out that for $(\underline{i}, \underline{k}) \in I(n,d) \times I(n,d)$,

$$\Delta(f_{\underline{i},\underline{k}}) = \sum_{\underline{j} \in I(n,d)} (-1)^{\partial_{\underline{i},\underline{j}} \partial_{\underline{j},\underline{k}}} \alpha(\varepsilon_{\underline{j},\underline{k}}; \varepsilon_{\underline{i},\underline{j}}) f_{\underline{i},\underline{j}} \otimes f_{\underline{j},\underline{k}}.$$

Hence, each $F(n,d)$ is a finite dimensional subcosuperalgebra of $F(n)$. Make the vector superspace V from the previous section into a right $F(n)$-cosupermodule with structure map $V \to V \otimes F(n)$ defined by

$$v_j \mapsto \sum_{i \in \{\pm 1, \dots, \pm n\}} (-1)^{\partial_i \partial_{i,j}} v_i \otimes f_{i,j}.$$

Then, for each $d \geq 1$, $V^{\otimes d}$ is also automatically a right $F(n)$-cosupermodule with structure map $V^{\otimes d} \to V^{\otimes d} \otimes F(n)$ given explicitly by the formula

$$v_{\underline{j}} \mapsto \sum_{\underline{i} \in I(n,d)} (-1)^{\partial_{\underline{i}} \partial_{\underline{i},\underline{j}}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i}}) v_{\underline{i}} \otimes f_{\underline{i},\underline{j}}.$$

In particular, $V^{\otimes d}$ can be viewed as a right $F(n,d)$-cosupermodule.

Let $E(n,d) = F(n,d)^*$ be the dual superalgebra. Let $e_{\underline{i},\underline{j}}$ denote the element of $E(n,d)$ with

$$e_{\underline{i},\underline{j}}(f_{\underline{i},\underline{j}}) = \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i},\underline{j}}), \qquad e_{\underline{i},\underline{j}}(f_{\underline{k},\underline{l}}) = 0 \text{ for } (\underline{k}, \underline{l}) \neq (\underline{i}, \underline{j}).$$

Then, the $\{e_{\underline{i},\underline{j}} \mid \underline{i}, \underline{j} \in I(n,d)\}$ give a basis for $E(n,d)$.

The right $F(n,d)$-cosupermodule $V^{\otimes d}$ is a left $E(n,d)$-supermodule in the way described above. Let $\rho_d : E(n,d) \to \mathrm{End}_{\Bbbk}(V^{\otimes d})$ be the resulting representation.

5.2. **Lemma.** *The representation ρ_d is an isomorphism between $E(n,d)$ and $\mathrm{End}_{\Bbbk}(V^{\otimes d})$. Moreover, $\rho_d(e_{\underline{i},\underline{j}}) = \dot{e}_{\underline{i},\underline{j}}$ for all $\underline{i}, \underline{j} \in I(n,d)$.*

Proof. It suffices to check that $e_{\underline{i},\underline{j}} v_{\underline{k}} = \dot{e}_{\underline{i},\underline{j}} v_{\underline{k}}$ for all $\underline{i}, \underline{j}, \underline{k} \in I(n,d)$. By the definition of the action of $E(n,d)$, we have that

$$\begin{aligned} e_{\underline{i},\underline{j}} v_{\underline{k}} &= (\mathrm{id} \,\bar{\otimes} e_{\underline{i},\underline{j}}) \left(\sum_{\underline{h} \in I(n,d)} (-1)^{\partial_{\underline{h}} \partial_{\underline{h},\underline{k}}} \alpha(\varepsilon_{\underline{h},\underline{k}}; \varepsilon_{\underline{h}}) v_{\underline{h}} \otimes f_{\underline{h},\underline{k}} \right) \\ &= \delta_{\underline{j},\underline{k}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i}}) \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i},\underline{j}}) v_{\underline{i}} = \delta_{\underline{j},\underline{k}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{j}}) v_{\underline{i}} = \dot{e}_{\underline{i},\underline{j}} v_{\underline{k}}. \end{aligned}$$

This completes the proof. $\square$

Now consider the superideal $\mathscr{I}(n)$ of $F(n)$ generated by the elements

$$\{f_{i,j} - f_{-i,-j}, f_{i,j}f_{k,l} - (-1)^{\partial_{i,j}\partial_{k,l}} f_{k,l}f_{i,j} \mid i,j,k,l = \pm 1, \dots, \pm n\}.$$

A short calculation reveals that this is actually a bisuperideal, so the quotient

$$B(n) := F(n)/\mathscr{I}(n)$$

is a bisuperalgebra quotient of $F(n)$. Let $b_{i,j} = f_{i,j} + \mathscr{I}(n)$. Then, $B(n)$ is just the free commutative superalgebra on the degree $\bar{0}$ generators $b_{i,j} = b_{-i,-j}$ and degree $\bar{1}$ generators $b_{i,-j} = b_{-i,j}$, for all $1 \leq i, j \leq n$. The superideal $\mathscr{I}(n)$ is homogeneous, so graded as $\mathscr{I}(n) = \bigoplus_{d \geq 0} \mathscr{I}(n,d)$. So $B(n)$ is also $\mathbb{Z}$-graded by degree as $B(n) = \bigoplus_{d \geq 0} B(n,d)$, with $B(n,d) \cong F(n,d)/\mathscr{I}(n,d)$. Moreover, $B(n,d)$ is spanned by all monomials $b_{\underline{i},\underline{j}} = b_{i_1,j_1} \dots b_{i_d,j_d}$ for $\underline{i}, \underline{j} \in I(n,d)$. The monomial $b_{\underline{i},\underline{j}}$ is non-zero if and only if $(\underline{i},\underline{j})$ is strict, and for strict $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l})$, we have that

$$b_{\underline{i},\underline{j}} = \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) b_{\underline{k},\underline{l}}.$$

It follows that $B(n,d)$ has basis $\{b_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in \Omega(n,d)\}$, where $\Omega(n,d)$ is the choice of W_d-orbit representatives in $I^2(n,d)$ made in the previous section.

Now, let $Q(n,d)$ denote the dual superalgebra $B(n,d)^*$. Since $B(n,d) = F(n,d)/\mathscr{I}(n,d)$, $Q(n,d)$ is naturally identified with the annihilator $\mathscr{I}(n,d)^\circ \subseteq E(n,d)$. For $(\underline{i},\underline{j}) \in I^2(n,d)$, let $\xi_{\underline{i},\underline{j}} \in Q(n,d) \subseteq E(n,d)$ denote the unique function with

$$\xi_{\underline{i},\underline{j}}(b_{\underline{i},\underline{j}}) = \alpha(\varepsilon_{\underline{i},\underline{j}};\varepsilon_{\underline{i},\underline{j}}), \qquad \text{and} \qquad \xi_{\underline{i},\underline{j}}(b_{\underline{k},\underline{l}}) = 0 \text{ for } (\underline{k},\underline{l}) \not\sim (\underline{i},\underline{j}).$$

The $\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in \Omega(n,d)\}$ give a basis for $Q(n,d)$.

We can regard the $F(n,d)$-cosupermodule $V^{\otimes d}$ instead as a $B(n,d)$-cosupermodule by restriction. Dualizing, we obtain a natural representation $Q(n,d) \to \mathrm{End}_{\Bbbk}(V^{\otimes d})$, which is nothing more than the restriction of the representation $\rho_d : E(n,d) \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(V^{\otimes d})$ defined earlier to the subsuperalgebra $Q(n,d) \subseteq E(n,d)$. Then:

5.3. **Theorem.** *The representation ρ_d gives an isomorphism between $Q(n,d)$ and the Schur superalgebra $\dot{Q}(n,d)$. Moreover, $\rho_d(\xi_{\underline{i},\underline{j}}) = \dot{\xi}_{\underline{i},\underline{j}}$ for all $(\underline{i},\underline{j}) \in I^2(n,d)$.*

Proof. Pick $(\underline{i},\underline{j}) \in I^2(n,d)$. Since $Q(n,d) \subseteq E(n,d)$, we can write

$$\xi_{\underline{i},\underline{j}} = \sum_{\underline{k},\underline{l} \in I(n,d)} a_{\underline{k},\underline{l}} e_{\underline{k},\underline{l}}$$

for coefficients $a_{\underline{k},\underline{l}} \in \Bbbk$. To calculate the coefficient $a_{\underline{k},\underline{l}}$, evaluate both sides at the element $f_{\underline{k},\underline{l}} \in F(n,d)$ to see that $a_{\underline{k},\underline{l}} \alpha(\varepsilon_{\underline{k},\underline{l}};\varepsilon_{\underline{k},\underline{l}}) = \xi_{\underline{i},\underline{j}}(f_{\underline{k},\underline{l}}) = \xi_{\underline{i},\underline{j}}(b_{\underline{k},\underline{l}})$. So by the definition of $\xi_{\underline{i},\underline{j}}$, $a_{\underline{k},\underline{l}}$ is zero unless $(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})$, in which case, $a_{\underline{k},\underline{l}} = \alpha(\varepsilon_{\underline{k},\underline{l}};\varepsilon_{\underline{k},\underline{l}}) \sigma(\underline{i},\underline{j};\underline{k},l) \xi_{\underline{i},\underline{j}}(b_{\underline{i},\underline{j}}) = \sigma(\underline{i},\underline{j};\underline{k},\underline{l})$. This shows that

$$\xi_{\underline{i},\underline{j}} = \sum_{(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})} \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) e_{\underline{k},\underline{l}}.$$

Now the theorem follows at once from Lemma 5.2, Theorem 4.5 and the definition (4.4). $\square$

We will henceforth *identify* $Q(n,d)$, which we defined as the dual of the cosuperalgebra $B(n,d)$, with $\dot{Q}(n,d)$, which we defined as the commutant of $W(d)$ on tensor space $V^{\otimes d}$. So the dual basis element $\xi_{\underline{i},\underline{j}} \in Q(n,d)$ is identified with the linear transformation $\dot{\xi}_{\underline{i},\underline{j}} \in \dot{Q}(n,d)$.

6 Weights and idempotents

Let $\Lambda(n,d)$ denote the set of all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers with $\lambda_1 + \dots + \lambda_n = d$. We partially order $\Lambda(n,d)$ by the usual *dominance order*, so $\lambda \geq \mu$ if and only if $\sum_{s=1}^t \lambda_s \geq \sum_{s=1}^t \mu_s$ for each $t = 1, \dots, n$. For $\underline{i} \in I(n,d)$, define its *weight* $\mathrm{wt}(\underline{i})$ to be the composition $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,d)$ where $\lambda_s = |\{t \mid 1 \leq t \leq d, |i_t| = s\}|$. Conversely, given $\lambda \in \Lambda(n,d)$, let $\underline{i}_\lambda$ denote the element $(1, \dots, 1, 2, \dots, 2, 3, \dots) \in I(n,d)$ where there are λ_1 ones, λ_2 twos, etc., so that $\mathrm{wt}(\underline{i}_\lambda) = \lambda$. Define

$$\xi_\lambda := \xi_{\underline{i}_\lambda, \underline{i}_\lambda} \in Q(n,d).$$

We call the elements $\{\xi_\lambda \mid \lambda \in \Lambda(n,d)\}$ *weight idempotents*, motivated by the following lemma:

6.1. **Lemma.** *For $(\underline{i},\underline{j}) \in I^2(n,d)$,*

$$\xi_\lambda \xi_{\underline{i},\underline{j}} = \begin{cases} \xi_{\underline{i},\underline{j}} & \textit{if } \mathrm{wt}(\underline{i}) = \lambda, \\ 0 & \textit{otherwise.} \end{cases} \qquad \xi_{\underline{i},\underline{j}} \xi_\lambda = \begin{cases} \xi_{\underline{i},\underline{j}} & \textit{if } \mathrm{wt}(\underline{j}) = \lambda, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, $\{\xi_\lambda \mid \lambda \in \Lambda(n,d)\}$ is a set of mutually orthogonal idempotents whose sum is the identity element of $Q(n,d)$.

Proof. It is elementary to check that the matrix units $\{e_{\underline{h},\underline{h}} \mid \underline{h} \in I(n,d)\}$ in $E(n,d)$ are a set of mutually orthogonal idempotents whose sum is the identity, with $e_{\underline{h},\underline{h}} e_{\underline{i},\underline{j}} = \delta_{\underline{h},\underline{i}} e_{\underline{i},\underline{j}}$ and $e_{\underline{i},\underline{j}} e_{\underline{h},\underline{h}} = \delta_{\underline{h},\underline{j}} e_{\underline{i},\underline{j}}$ for all $\underline{h}, \underline{i}, \underline{j} \in I(n,d)$. Now according to (4.4), $\xi_\lambda = \sum_{\underline{h}} e_{\underline{h},\underline{h}}$ summing over all $\underline{h} \in I(n,d)$ with $\mathrm{wt}(\underline{h}) = \lambda$, as an element of $E(n,d)$. The lemma follows easily from these remarks. $\square$

Let ω denote the weight (1^d), which is an element of $\Lambda(n,d)$ providing $n \geq d$. Assuming this, the weight idempotent ξ_ω is a well-defined element of $Q(n,d)$, and $\xi_\omega Q(n,d) \xi_\omega$ is naturally a superalgebra in its own right, its identity element being the idempotent ξ_ω. We have the following double centralizer property:

6.2. **Theorem.** *Assume that $n \geq d$.*

(i) *The map $\phi : Q(n,d)\xi_\omega \to V^{\otimes d}$, $\xi_{\underline{i},\underline{i}_\omega} \mapsto v_{\underline{i}}$ for $\underline{i} \in I(n,d)$ is an even isomorphism of $Q(n,d)$-supermodules. In particular, $V^{\otimes d}$ is a projective $Q(n,d)$-supermodule.*

(ii) *The map $\psi : W(d) \to \xi_\omega Q(n,d) \xi_\omega$, $x \otimes c^\delta \mapsto \xi_{\underline{i}_\omega \cdot (x,\delta), \underline{i}_\omega}$ for all $(x,\delta) \in W_d$, is a superalgebra isomorphism.*

(iii) $\mathrm{End}_{Q(n,d)}(V^{\otimes d}) \cong W(d)$.

Proof. For (i), we first claim that $\xi_{\underline{i},\underline{i}_\omega} v_{\underline{i}_\omega} = v_{\underline{i}}$. Well, $\xi_{\underline{i},\underline{i}_\omega} = \sum_{(\underline{k},\underline{l})\sim(\underline{i},\underline{i}_\omega)} e_{\underline{k},\underline{l}}$, and $e_{\underline{k},\underline{l}} v_{\underline{i}_\omega} = \delta_{\underline{l},\underline{i}_\omega} v_{\underline{k}}$. Now observe that $(\underline{k},\underline{i}_\omega) \sim (\underline{i},\underline{i}_\omega)$ if and only if $\underline{k} = \underline{i}$, since $\mathrm{Stab}_{W_d}(\underline{i}_\omega) = 1$. It now follows easily that $\xi_{\underline{i},\underline{i}_\omega} v_{\underline{i}_\omega} = v_{\underline{i}}$ as claimed. So in particular, $\xi_\omega v_{\underline{i}_\omega} = v_{\underline{i}_\omega}$, so there is a well-defined $Q(n,d)$-module homomorphism $Q(n,d)\xi_\omega \to V^{\otimes d}$ such that $\xi_\omega \mapsto v_{\underline{i}_\omega}$. By the claim, this is precisely the map ϕ. Finally, observe that $Q(n,d)\xi_\omega$ has as basis the elements $\{\xi_{\underline{i},\underline{i}_\omega} \mid \underline{i} \in I(n,d)\}$, so that ϕ is an isomorphism.

For (ii) and (iii), ξ_ω is an idempotent, so the superalgebras $\mathrm{End}_{Q(n,d)}(Q(n,d)\xi_\omega)$ and $\xi_\omega Q(n,d)\xi_\omega$ are naturally isomorphic. There is a homomorphism $W(d) \to \mathrm{End}_{Q(n,d)}(V^{\otimes d})$ defined by the representation of $W(d)$ on $V^{\otimes d}$. Combining these with (i), we obtain a superalgebra homomorphism $\psi : W(d) \to \xi_\omega Q(n,d)\xi_\omega$. By definition, it maps the element $x \otimes c^\delta \in W(d)$ to the unique element ξ of $\xi_\omega Q(n,d)\xi_\omega$ with $\xi\phi = v_{\underline{i}_\omega}(x \otimes c^\delta)$. But $v_{\underline{i}_\omega}(x \otimes c^\delta) = v_{\underline{i}_\omega \cdot (x,\delta)}$, so $\psi(x \otimes c^\delta) = \xi_{\underline{i}_\omega \cdot (x,\delta), \underline{i}_\omega}$ as in the lemma. It remains to observe that the elements $\{\xi_{\underline{i}_\omega \cdot (x,\delta), \underline{i}_\omega} \mid (x,\delta) \in W_d\}$ give a basis for $\xi_\omega Q(n,d)\xi_\omega$, so that ψ is an isomorphism. $\square$

Using Theorem 6.2(ii), Corollary 2.13, Lemma 3.2 and Corollary 3.5, we deduce:

6.3. **Lemma.** *For $n \geq d$, the number of irreducible $Q(n,d)$-supermodules not annihilated by ξ_ω is equal to $|\mathscr{RP}_p(d)|$.*

There is one other situation where Schur functors arising from weight idempotents will be useful. Suppose now that $m \geq n$. We embed $\Lambda(n,d)$ into $\Lambda(m,d)$ as the set of all weights of the form $(\lambda_1, \dots, \lambda_n, 0, \dots, 0)$, and $I(n,d)$ into $I(m,d)$ as the set of all $\underline{i} \in I(m,d)$ with $i_s \in \{\pm 1, \dots, \pm n\}$ for each $s = 1, \dots, d$. To avoid confusion with the corresponding elements of $Q(n,d)$, we denote the elements $\xi_\lambda, \xi_{\underline{i},\underline{j}} \in Q(m,d)$ for $\lambda \in \Lambda(m,d), (\underline{i},\underline{j}) \in I^2(m,d)$ instead by $\widehat{\xi}_\lambda, \widehat{\xi}_{\underline{i},\underline{j}}$ respectively. Let $e \in Q(m,d)$ denote the idempotent

$$e = \sum_{\lambda \in \Lambda(n,d) \subseteq \Lambda(m,d)} \widehat{\xi}_\lambda. \tag{6.4}$$

If $\underline{i},\underline{j} \in I(n,d) \subseteq I(m,d)$, the element $\widehat{\xi}_{\underline{i},\underline{j}} \in Q(m,d)$ lies in $eQ(m,d)e$.

6.5. **Lemma.** *The map $\iota : Q(n,d) \to eQ(m,d)e$, $\xi_{\underline{i},\underline{j}} \mapsto \widehat{\xi}_{\underline{i},\underline{j}}$ for all $(\underline{i},\underline{j}) \in I^2(n,d)$, is a superalgebra isomorphism.*

Proof. Consider the $\mathbb{Z}$-graded superideal $\mathscr{J}(m) = \bigoplus_{d \geq 0} \mathscr{J}(m,d)$ of $B(m)$ generated by the elements

$$\{b_{i,j} \mid i \text{ or } j \text{ equals } \pm(n+1), \pm(n+2), \dots, \pm m\}.$$

One checks easily that $\Delta(\mathscr{J}(m)) \subseteq \mathscr{J}(m) \otimes B(m) + B(m) \otimes \mathscr{J}(m)$, so that the comultiplication Δ on $B(m)$ induces a well-defined comultiplication on $B(m)/\mathscr{J}(m)$ (though $\mathscr{J}(m)$ is *not* a cosuperideal). Evidently, $B(m)/\mathscr{J}(m) \cong B(n)$ as superalgebras, the induced comultiplication on $B(m)/\mathscr{J}(m)$ corresponding to the usual comultiplication on $B(n)$ under the isomorphism. Dualizing, we obtain a *multiplicative* even isomorphism between $Q(n,d)$ and $\mathscr{J}(m)^\circ \subseteq eQ(m,d)e$, being precisely the map ι. Finally, observe that $eQ(m,d)e = \mathscr{J}(m)^\circ$ to complete the proof. $\square$

Next, we introduce a subsuperalgebra of $Q(n,d)$ which plays the role of Cartan subalgebra. Let $\mathscr{J}_0(n) = \bigoplus_{d\geq 0} \mathscr{J}_0(n,d)$ denote the $\mathbb{Z}$-graded superideal of $B(n)$ generated by the elements

$$\{b_{i,j} \mid i,j = \pm 1, \dots, \pm n, |i| \neq |j|\}.$$

It is elementary to check that $\mathscr{J}_0(n)$ is a bisuperideal of $B(n)$, so we can form the bisuperalgebra quotient $B_0(n) := B(n)/\mathscr{J}_0(n)$. For $i = 1, \dots, n$, let x_i denote the image of $b_{i,i} = b_{-i,-i}$ in $B_0(n)$, and $\bar{x}_i$ denote the image of $b_{i,-i} = b_{-i,i}$. Then $B_0(n)$ is precisely the free commutative superalgebra on the generators $x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$. Comultiplication $\Delta : B_0(n) \to B_0(n) \otimes B_0(n)$ is given explicitly on these generators by

$$\Delta(x_i) = x_i \otimes x_i - \bar{x}_i \otimes \bar{x}_i, \qquad \Delta(\bar{x}_i) = x_i \otimes \bar{x}_i + \bar{x}_i \otimes x_i.$$

As usual, $B_0(n)$ is $\mathbb{Z}$-graded by degree as $\bigoplus_{d\geq 0} B_0(n,d)$, with $B_0(n,d) \cong B(n,d)/\mathscr{J}_0(n,d)$ being a subsupercoalgebra of $B_0(n)$ for each $d \geq 0$. The dual superalgebra $Q_0(n,d) = B_0(n,d)^*$ can be identified with the annihilator $\mathscr{J}_0(n,d)^\circ \subseteq Q(n,d)$, giving us a subsuperalgebra of $Q(n,d)$.

Consider the special case $Q_0(1,d)$ for $d \geq 1$ in more detail (obviously, $Q_0(1,0) = \mathbb{k}$). Writing $x = x_1, \bar{x} = \bar{x}_1$, the elements $\{x^d, x^{d-1}\bar{x}\}$ give a basis for $B_0(1,d)$, with comultiplication $\Delta : B_0(1,d) \to B_0(1,d) \otimes B_0(1,d)$ satisfying

$$\Delta(x^d) = x^d \otimes x^d - dx^{d-1}\bar{x} \otimes x^{d-1}\bar{x}, \qquad \Delta(x^{d-1}\bar{x}) = x^{d-1}\bar{x} \otimes x^d + x^d \otimes x^{d-1}\bar{x}.$$

As a basis for $Q_0(1,d)$, take the dual basis $\{y_d, \bar{y}_d\}$ to the basis $\{x^d, x^{d-1}\bar{x}\}$ of $B_0(1,d)$. The superalgebra multiplication, dual to the comultiplication in $B_0(1,d)$, is then given by $y_d y_d = y_d, y_d \bar{y}_d = \bar{y}_d = \bar{y}_d y_d, \bar{y}_d \bar{y}_d = d y_d$. Hence, for $d \geq 1$,

$$Q_0(1,d) \cong \begin{cases} C(1) & \text{if } p \nmid d, \\ \bigwedge(1) & \text{if } p \mid d, \end{cases}$$

recalling Example 2.2.

Now in general, the subsuperalgebra $Q_0(n,d) \subseteq Q(n,d)$ contains each weight idempotent ξ_λ for $\lambda \in \Lambda(n,d)$ in its center. So,

$$Q_0(n,d) \cong \prod_{\lambda \in \Lambda(n,d)} \xi_\lambda Q_0(n,d). \tag{6.6}$$

Moreover, one can see that

$$\xi_\lambda Q_0(n,d) \cong Q_0(1,\lambda_1) \otimes \cdots \otimes Q_0(1,\lambda_n) \cong C(h_{p'}(\lambda)) \otimes \bigwedge(h_p(\lambda)) \tag{6.7}$$

where $h_p(\lambda)$ denotes the number of non-zero parts of λ that are divisible by p, and $h_{p'}(\lambda)$ denotes the number of parts of λ that are coprime to p. We deduce immediately using Lemma 2.9, Example 2.7 and Example 2.10 that $\xi_\lambda Q_0(n,d)$ has a unique irreducible supermodule up to isomorphism, of dimension $2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}$. We pick one such and denote by $U(\lambda)$. Note $U(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even. Finally, regarding $U(\lambda)$ as an $Q_0(n,d)$-supermodule by inflation, we have shown:

6.8. **Lemma.** *The supermodules $\{U(\lambda) \mid \lambda \in \Lambda(n,d)\}$ give a complete set of pairwise non-isomorphic irreducible $Q_0(n,d)$-supermodules. The dimension of $U(\lambda)$ is $2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}$, and $U(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

Recalling Lemma 5.1, we have thus determined the irreducible $B_0(n)$-cosupermodules, namely, the $B_0(n)$-cosupermodules $\{U(\lambda) \mid \lambda \in \Lambda(n)\}$, where $\Lambda(n) := \bigcup_{d \geq 0} \Lambda(n,d)$. Now let M be an arbitrary finite dimensional $B(n)$-cosupermodule with structure map $\eta : M \to M \otimes B(n)$. By Lemma 5.1, M decomposes as $M = \bigoplus_{d \geq 0} M_d$ where M_d is the largest subcosupermodule with $\eta(M_d) \subseteq M_d \otimes B(n,d)$. Each M_d is naturally a $B(n,d)$-cosupermodule, hence a $Q(n,d)$-supermodule. Then, for $\lambda \in \Lambda(n,d)$, we define the λ-*weight space* of M to be the space $M_\lambda := \xi_\lambda M_d$. Recalling (6.6), M_λ is a $Q_0(n,d)$-subsupermodule of M_d. Equivalently, M_λ is a $B_0(n)$-subcosupermodule of M, viewing M as a $B_0(n)$-cosupermodule by restriction, and

$$M = \bigoplus_{\lambda \in \Lambda(n)} M_\lambda.$$

Let $X(n)$ denote the free polynomial algebra $\mathbb{Z}[x_1, \dots, x_n]$ and for $\lambda \in \Lambda(n)$, set $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. Define the *formal character*

$$\operatorname{ch} M = \sum_{\lambda \in \Lambda(n)} \dim M_\lambda x^\lambda \in X(n).$$

Note that for finite dimensional $B(n)$-cosupermodules M, N, we have that $\operatorname{ch}(M \oplus N) = \operatorname{ch} M + \operatorname{ch} N$ and $\operatorname{ch}(M \otimes N) = \operatorname{ch} M . \operatorname{ch} N$. In other words, the map $\operatorname{ch} : \operatorname{Grot}(B(n)) \to X(n)$ is a ring homomorphism from the Grothendieck ring of the category of finite dimensional right $B(n)$-cosupermodules to $X(n)$.

7 The big cell

Let $\mathscr{I}_\flat(n) = \bigoplus_{d \geq 0} \mathscr{I}_\flat(n,d)$ and $\mathscr{I}_\sharp(n) = \bigoplus_{d \geq 0} \mathscr{I}_\sharp(n,d)$ denote the $\mathbb{Z}$-graded superideals of $B(n)$ generated by the elements

$$\{b_{i,j} \mid i,j = \pm 1, \dots, \pm n, |i| < |j|\}, \quad \{b_{i,j} \mid i,j = \pm 1, \dots, \pm n, |i| > |j|\}$$

respectively. One easily checks that these are cosuperideals. Hence, we can form the bisuperalgebras quotients

$$B_\flat(n) := B(n)/\mathscr{I}_\flat(n), \qquad B_\sharp(n) := B(n)/\mathscr{I}_\sharp(n).$$

Both $B_\flat(n)$ and $B_\sharp(n)$ are $\mathbb{Z}$-graded with degree d component, denoted $B_\flat(n,d)$ and $B_\sharp(n,d)$ respectively, being cosuperalgebra quotients of $B(n,d)$. The corresponding dual superalgebras to these, namely $Q_\flat(n,d) = \mathscr{I}_\flat(n,d)^\circ$ and $Q_\sharp(n,d) = \mathscr{I}_\sharp(n,d)^\circ$, are therefore subsuperalgebras of $Q(n,d)$, called the negative Borel and positive Borel subsuperalgebras respectively. They are spanned by the elements

$$\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \geq |\underline{j}|\} \quad \text{and} \quad \{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \leq |\underline{j}|\}$$

respectively, where $|\underline{i}| \geq |\underline{j}|$ means that $|i_k| \geq |j_k|$ for each $k = 1, \dots, d$. Let $\pi_\flat : B(n) \to B_\flat(n)$ and $\pi_\sharp : B(n) \to B_\sharp(n)$ denote the natural quotient maps and set $b^\flat_{\underline{i},\underline{j}} = \pi_\flat(b_{\underline{i},\underline{j}})$, $b^\sharp_{\underline{i},\underline{j}} = \pi_\sharp(b_{\underline{i},\underline{j}})$ for $\underline{i}, \underline{j} \in I(n,d)$. In particular, $b^\flat_{\underline{i},\underline{j}} = 0$ unless $|\underline{i}| \geq |\underline{j}|$ and $b^\sharp_{\underline{i},\underline{j}} = 0$ unless $|\underline{i}| \leq |\underline{j}|$. Let

$$\pi : B(n) \to B_\flat(n) \otimes B_\sharp(n)$$

be the map $(\pi_\flat \otimes \pi_\sharp) \circ \Delta$. We wish to prove that this map π is injective, this being an analogue of the existence of the big cell in reductive algebraic groups, crucial for highest weight theory. It is possible to give a quick proof in the setting of algebraic supergroups. Since we wish to avoid introducing this language, we content ourselves with an elementary direct proof, though it is rather lengthy:

7.1. **Theorem.** *π is injective.*

Proof. We proceed in a number of steps. Observe right away that it is enough to prove that π is injective on each $B(n,d)$ separately. So, fix $d \geq 1$ and consider the restriction $\pi : B(n,d) \to B_\flat(n,d) \otimes B_\sharp(n,d)$. Let

$$Y = \{(\underline{i}, \underline{k}, \underline{l}, \underline{j}) \in I(n,d) \times I(n,d) \times I(n,d) \times I(n,d) \mid |\underline{i}| \geq |\underline{k}|, |\underline{l}| \leq |\underline{j}|\}.$$

Write $(\underline{i}, \underline{k}, \underline{l}, \underline{j}) \approx (\underline{i}', \underline{k}', \underline{l}', \underline{j}')$ if both $(\underline{i}, \underline{k}) \sim (\underline{i}', \underline{k}')$ and $(\underline{l}, \underline{j}) \sim (\underline{l}', \underline{j}')$. Also call $(\underline{i}, \underline{k}, \underline{l}, \underline{j})$ *strict* if both $(\underline{i}, \underline{k})$ and $(\underline{l}, \underline{j})$ are strict in the sense of Lemma 4.2. Then:

7.2. *If Z is a choice of representatives for the $\approx$-equivalence classes of strict $(\underline{i}, \underline{k}, \underline{l}, \underline{j}) \in Y$, then $\{b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{l},\underline{j}} \mid (\underline{i}, \underline{k}, \underline{l}, \underline{j}) \in Z\}$ is a basis for $B_\flat(n,d) \otimes B_\sharp(n,d)$.*

Now define $\underline{m}(\underline{i}, \underline{j})$, for any $\underline{i}, \underline{j} \in I(n,d)$, to be the unique element $\underline{m} \in I(n,d)$ with

$$m_s = \begin{cases} i_s & \text{if } |i_s| < |j_s| \\ j_s & \text{if } |i_s| \geq |j_s| \end{cases}$$

for all $s = 1, \dots, d$. Observe that $\underline{m}(\underline{i} \cdot g, \underline{j} \cdot g) = \underline{m}(\underline{i}, \underline{j}) \cdot g$ for all $g \in W_d$. We claim:

7.3. *Suppose $\underline{i}, \underline{j} \in I(n,d)$ and $g \in W_d$ are such that $\underline{m}(\underline{i}, \underline{j}) = \underline{m}(\underline{i}, \underline{j} \cdot g) = \underline{m}(\underline{i} \cdot g, \underline{j} \cdot g)$. Then, $(\underline{i}, \underline{j}) \sim (\underline{i}, \underline{j} \cdot g)$.*

We prove (7.3) by induction on d. Let $\underline{m} = \underline{m}(\underline{i}, \underline{j})$. If $d = 1$, then the assumption that $\underline{m} \cdot g = \underline{m}$ forces $g = 1$, and the lemma follows trivially. Now suppose that $d > 1$ and that we have proved (7.3) for all smaller d. Write $\{\pm 1, \dots, \pm d\} = I \sqcup J$ where

$$I = \{\pm s \mid 1 \leq s \leq d, |i_s| \geq |j_s|\},$$
$$J = \{\pm s \mid 1 \leq s \leq d, |i_s| < |j_s|\}.$$

Suppose first that g stabilizes I. Then, we can write $g = xy$ where x fixes J pointwise and y fixes I pointwise. The assumption that $\underline{m} = \underline{m} \cdot g$ implies that both $\underline{m} = \underline{m} \cdot x$ and $\underline{m} = \underline{m} \cdot y$. For $s \in J$, $m_s = i_s$ and $m_{ys} = i_{ys}$, so since $m_s = m_{ys}$, we see that $i_s = i_{ys}$. Hence $\underline{i} \cdot y = \underline{i}$, and a similar argument gives that $\underline{j} \cdot x = \underline{j}$. So, $(\underline{i}, \underline{j} \cdot g) = (\underline{i} \cdot y, \underline{j} \cdot y) \sim (\underline{i}, \underline{j})$ as required.

Now suppose that g does not stabilize I. Then, we can pick $s \in I$ such that $gs \in J$. Let $t = gs \in J$ and define x to be the unique element of W_d with $xs = t, xt = s$ and fixing all other elements of $\{\pm 1, \dots, \pm d\} \setminus \{\pm s, \pm t\}$. Set $g' = xg, \underline{j}' = \underline{j} \cdot x$, so $\underline{j}' \cdot g' = \underline{j}g$. Using that $\underline{m} \cdot g = \underline{m}$, we have that $j_s = m_s = m_t = i_t$. So, $|j_t| > |i_t| = |m_t| = |m_s|$. Using $\underline{m} = \underline{m}(\underline{i}, \underline{j} \cdot g)$, we must therefore have that $m_s = i_s = i_t = m_t$. This shows that $\underline{i} \cdot x = \underline{i}$ and $\underline{m} \cdot x = \underline{m}$. Now,

$$\begin{aligned} \underline{m}(\underline{i}, \underline{j}) = \underline{m}(\underline{i} \cdot x, \underline{j} \cdot x) &= \underline{m}(\underline{i}, \underline{j}'), \\ \underline{m}(\underline{i}, \underline{j} \cdot g) &= \underline{m}(\underline{i}, \underline{j}' \cdot g'), \\ \underline{m}(\underline{i} \cdot g, \underline{j} \cdot g) &= \underline{m}(\underline{i} \cdot g', \underline{j}' \cdot g'). \end{aligned}$$

So by our assumption, $\underline{m}(\underline{i}, \underline{j}') = \underline{m}(\underline{i}, \underline{j}' \cdot g') = \underline{m}(\underline{i} \cdot g', \underline{j}' \cdot g')$. Now, $g's = s$, so we deduce by induction that $(\underline{i}, \underline{j}') \sim (\underline{i}, \underline{j}' \cdot g')$. Hence, $(\underline{i}, \underline{j}) \sim (\underline{i} \cdot x, \underline{j} \cdot x) = (\underline{i}, \underline{j}') \sim (\underline{i}, \underline{j}' \cdot g') = (\underline{i}, \underline{j} \cdot g)$ as required to complete the proof of (7.3).

Now we apply (7.3) to show:

7.4. *Let $\underline{i}, \underline{j}, \underline{i}', \underline{j}' \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i}, \underline{j})$, $\underline{m}' = \underline{m}(\underline{i}', \underline{j}')$. If $(\underline{i}, \underline{m}, \underline{m}, \underline{j}) \approx (\underline{i}', \underline{m}', \underline{m}', \underline{j}')$ then $(\underline{i}, \underline{j}) \sim (\underline{i}', \underline{j}')$.*

Indeed, take $g, h \in W_d$ such that $(\underline{i}, \underline{m}) = (\underline{i}' \cdot g, \underline{m}' \cdot g)$ and $(\underline{m}, \underline{j}) = (\underline{m}' \cdot gh, \underline{j}' \cdot gh)$. Set $\underline{k} = \underline{j}' \cdot g$. Now,

$$\begin{aligned} \underline{m} = \underline{m}(\underline{i}, \underline{j}) = \underline{m}(\underline{i}, \underline{j}' \cdot gh) &= \underline{m}(\underline{i}, \underline{k} \cdot h), \\ \underline{m}' \cdot g = \underline{m}(\underline{i}' \cdot g, \underline{j}' \cdot g) &= \underline{m}(\underline{i}, \underline{k}), \\ \underline{m}' \cdot gh = \underline{m}(\underline{i}' \cdot gh, \underline{j}' \cdot gh) &= \underline{m}(\underline{i} \cdot h, \underline{k} \cdot h). \end{aligned}$$

So, observing that $\underline{m} = \underline{m}' \cdot g = \underline{m}' \cdot gh$, we have that $\underline{m}(\underline{i}, \underline{k}) = \underline{m}(\underline{i}, \underline{k} \cdot h) = \underline{m}(\underline{i} \cdot h, \underline{k} \cdot h)$. Hence by (7.3), $(\underline{i}, \underline{k}) \sim (\underline{i}, \underline{k} \cdot h)$. So $(\underline{i}', \underline{j}') \sim (\underline{i}' \cdot g, \underline{j}' \cdot g) = (\underline{i}, \underline{k}) \sim (\underline{i}, \underline{k} \cdot h) = (\underline{i}, \underline{j})$.

Next we claim:

7.5. *Let $\underline{i}, \underline{j} \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i}, \underline{j})$. If $(\underline{i}, \underline{j})$ is strict, then $(\underline{i}, \underline{m}, \underline{m}, \underline{j})$ is strict.*

To prove this, take $(\underline{i}, \underline{j})$ strict and suppose that $(\underline{i}, \underline{m})$ is not strict. Then, there exist $1 \le s < t \le d$ with $|i_s| = |i_t|, |m_s| = |m_t|$ and $\partial_{i_s, m_s} \partial_{i_t, m_t} = \bar{1}$. So, $i_s \neq m_s, i_t \neq m_t$, hence by the definition of $\underline{m}$, $m_s = j_s, m_t = j_t$. But this contradicts the fact that $(\underline{i}, \underline{j})$ is strict. Hence, $(\underline{i}, \underline{m})$ is strict, and a similar argument shows that $(\underline{m}, \underline{j})$ is strict.

Recall that $\Omega(n,d)$ is some set of representatives of the $\sim$-equivalence classes of strict $(\underline{i}, \underline{j}) \in I(n,d) \times I(n,d)$. In view of (7.4) and (7.5), all $\{(\underline{i}, \underline{m}, \underline{m}, \underline{j}) \mid (\underline{i}, \underline{j}) \in \Omega(n,d), \underline{m} = \underline{m}(\underline{i}, \underline{j})\}$ are strict and lie in different $\approx$-equivalence classes. So they are linearly independent by (7.2), and we have now proved:

7.6. *The elements $\{b^\flat_{\underline{i},\underline{m}} \otimes b^\sharp_{\underline{m},\underline{j}} \mid (\underline{i}, \underline{j}) \in \Omega(n,d), \underline{m} = \underline{m}(\underline{i}, \underline{j})\}$ are linearly independent.*

Now we can prove the theorem. Call $(\underline{i},\underline{k},\underline{l},\underline{j}) \in Y$ *special* if there exists $g \in W_d$ such that

$$i_{gs} = k_{gs} = l_s \text{ whenever } |l_s| < |j_s|,$$
$$l_s = j_s = k_{gs} \text{ whenever } |l_s| = |j_s|$$

for all $s = 1, \dots, d$. We point out that if $\underline{m} = \underline{m}(\underline{i},\underline{j})$, then $(\underline{i},\underline{m},\underline{m},\underline{j})$ is special. Now, if $(\underline{i},\underline{k},\underline{l},\underline{j}) \approx (\underline{i}',\underline{k}',\underline{l}',\underline{j}')$ and $(\underline{i},\underline{k},\underline{l},\underline{j})$ is special, then $(\underline{i}',\underline{k}',\underline{l}',\underline{j}')$ is too. So the property of being special is a property of $\approx$-equivalence classes. Choose a total order $\succ$ on the set of all special $\approx$-equivalence classes such that the following hold for all special $(\underline{i},\underline{k},\underline{l},\underline{j}), (\underline{i}',\underline{k}',\underline{l}',\underline{j}') \in Y$:

(1) if $\mathrm{wt}(\underline{k}') > \mathrm{wt}(\underline{k})$ (in the dominance order) then $(\underline{i}',\underline{k}',\underline{l}',\underline{j}') \succ (\underline{i},\underline{k},\underline{l},\underline{j})$;

(2) if $\mathrm{wt}(\underline{k}) = \mathrm{wt}(\underline{k}')$ and $|\{s \mid 1 \le s \le d, i_s = k_s\}| > |\{s \mid 1 \le s \le d, i'_s = k'_s\}|$ then $(\underline{i}',\underline{k}',\underline{l}',\underline{j}') \succ (\underline{i},\underline{k},\underline{l},\underline{j})$.

We need one more claim:

7.7. *Let $\underline{i},\underline{j} \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i},\underline{j})$. Then,*

$$\pi(b_{\underline{i},\underline{j}}) = \pm b^\flat_{\underline{i},\underline{m}} \otimes b^\sharp_{\underline{m},\underline{j}} + A + B$$

where A is a linear combination of terms of the form $b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}}$ with $(\underline{i},\underline{k},\underline{k},\underline{j})$ special and $(\underline{i},\underline{k},\underline{k},\underline{j}) \succ (\underline{i},\underline{m},\underline{m},\underline{j})$, and B is a linear combination of terms of the form $b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}}$ with $(\underline{i},\underline{k},\underline{k},\underline{j})$ not special.

To prove (7.7), we have from the definition of π that

$$\pi(b_{i,j}) = \pm b^\flat_{i,m} \otimes b^\sharp_{m,j} \pm b^\flat_{i,-m} \otimes b^\sharp_{-m,j} + (\text{a linear combination of } b^\flat_{i,k} \otimes b^\sharp_{k,j} \text{ with } |k| < |m|)$$

where $m = \min(|i|,|j|)$. So, writing $\underline{m} = \underline{m}(\underline{i},\underline{j})$,

$$\pi(b_{\underline{i},\underline{j}}) = \sum_{\delta \in \mathbb{Z}_2^d} \pm b^\flat_{\underline{i},\underline{m}\cdot\delta} \otimes b^\sharp_{\underline{m}\cdot\delta,\underline{j}} + (\text{a linear combination of } b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}} \text{ with } \mathrm{wt}(\underline{k}) > \mathrm{wt}(\underline{m}).)$$

Therefore, we just need to show that for all $(\bar{0},\bar{0},\dots,\bar{0}) \neq \delta \in \mathbb{Z}_2^d$ such that $(\underline{i},\underline{m}\cdot\delta,\underline{m}\cdot\delta,\underline{j})$ is special, we have that $|\{s \mid 1 \le s \le d, i_s = m_s\}| > |\{s \mid 1 \le s \le d, i_s = m_{\delta s}\}|$. Take $\delta \in \mathbb{Z}_2^d$ such that $(\underline{i},\underline{m}\cdot\delta,\underline{m}\cdot\delta,\underline{j})$ is special. Then certainly we have that $m_{\delta s} = j_s$ whenever $|m_s| = |j_s|$, when $m_s = j_s$ by definition of $\underline{m}$. So for s with $|m_s| = |j_s|$, we have that $m_{\delta s} = m_s$, whence $\delta_s = \bar{0}$. Instead, take t with $|m_t| < |j_t|$. Then, $m_t = i_t$ so $m_t = i_{\delta t}$ if and only if $\delta_t = \bar{0}$. These observations establish that

$$|\{s \mid 1 \le s \le d, i_s = m_s\}| \ge |\{s \mid 1 \le s \le d, i_s = m_{\delta s}\}|$$

with equality if and only if $\delta = (\bar{0},\bar{0},\dots,\bar{0})$. This completes the proof of (7.7).

Now the theorem follows easily from (7.6), (7.7) and a unitriangular argument involving the order $\succ$. $\square$

7.8. **Corollary.** *The natural multiplication map $\mu : Q_\flat(n,d) \otimes Q_\sharp(n,d) \to Q(n,d)$ is surjective.*

8 Highest weight theory

Now we can classify the irreducible $Q(n,d)$-supermodules using highest weight theory. Recall that $Q_\sharp(n,d)$ denotes the positive Borel subsuperalgebra of $Q(n,d)$. We begin by determining the irreducible $Q_\sharp(n,d)$-supermodules.

The superideal $\mathscr{I}_\sharp(n)$ from §7 is contained in the superideal $\mathscr{I}_0(n)$ from §6. It follows that $Q_0(n,d) \subseteq Q_\sharp(n,d)$. On the other hand, let $Q_+(n,d)$ denote the subsuperspace of $Q_\sharp(n,d)$ spanned by the elements

$$\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \leq |\underline{j}|, |i_s| < |j_s| \text{ for some } s\}.$$

It follows from Lemma 6.1 that $Q_+(n,d)$ is a superideal of $Q_\sharp(n,d)$. Moreover, $Q_\sharp(n,d) = Q_0(n,d) \oplus Q_+(n,d)$ as a vector superspace, and $Q_\sharp(n,d)/Q_+(n,d) \cong Q_0(n,d)$. Analogously, $Q_-(n,d)$ denotes the superideal spanned by the elements $\{\xi_{\underline{i},\underline{j}} | (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \geq |\underline{j}|, |i_s| > |j_s| \text{ for some } s\}$, and $Q_\flat(n,d) = Q_0(n,d) \oplus Q_-(n,d)$.

If M is any $Q_0(n,d)$-supermodule, we can view it as a $Q_\sharp(n,d)$-supermodule by inflation along the quotient map $Q_\sharp(n,d) \twoheadrightarrow Q_0(n,d)$. In particular, we obtain irreducible $Q_\sharp(n,d)$-modules denoted $\{U(\lambda) \mid \lambda \in \Lambda(n,d)\}$, namely, the inflations of the irreducible $Q_0(n,d)$-supermodules constructed in Lemma 6.8.

Now suppose that M is a non-zero $Q_\sharp(n,d)$-supermodule and $\lambda \in \Lambda(n,d)$. By Lemma 6.1, for $\xi \in Q_+(n,d)$, $\xi M_\lambda \subseteq \bigoplus_{\mu > \lambda} M_\mu$. It follows at once that for any weight λ maximal in the dominance order such that $M_\lambda \neq 0$ (such a weight certainly exists as there are finitely many weights!), the weight space M_λ is annihilated by $Q_+(n,d)$. So M_λ is a $Q_\sharp(n,d)$-subsupermodule of M and the action of $Q_\sharp(n,d)$ on M_λ factors through the quotient $Q_0(n,d)$. In particular, if M is an irreducible $Q_\sharp(n,d)$-supermodule, $M \cong U(\lambda)$.

Given an arbitrary weight λ, we call a $Q(n,d)$-supermodule M a *highest weight module* of *highest weight* λ if the following conditions hold:

(1) M_λ is a $Q_\sharp(n,d)$-subsupermodule of M isomorphic to $U(\lambda)$;

(2) M is generated as an $Q(n,d)$-supermodule by M_λ.

For $\lambda \in \Lambda(n,d)$, define

$$V(\lambda) := Q(n,d) \otimes_{Q_\sharp(n,d)} U(\lambda). \tag{8.1}$$

Call the weight λ an *admissible weight* if $V(\lambda) \neq 0$.

8.2. **Lemma.** *For admissible λ, $V(\lambda)$ is a highest weight module of highest weight λ. Moreover, $V(\lambda)_\mu = 0$ unless $\mu \leq \lambda$.*

Proof. Recalling Corollary 7.8, we certainly have that

$$V(\lambda) = Q_\flat(n,d) \otimes U(\lambda) = Q_-(n,d) \otimes U(\lambda) \oplus Q_0(n,d) \otimes U(\lambda).$$

All weights of $Q_-(n,d) \otimes U(\lambda)$ are strictly lower than λ in the dominance order. So the λ-weight space of $V(\lambda)$ is equal to $1 \otimes U(\lambda)$, a homomorphic image of $U(\lambda)$. The assumption that λ is admissible is equivalent to $1 \otimes U(\lambda)$ being non-zero, in which case it is isomorphic to $U(\lambda)$ as $U(\lambda)$ is irreducible. □

The admissible $V(\lambda)$ have the following universal property:

8.3. **Lemma.** *Suppose that M is a highest weight module of highest weight λ. Then, λ is admissible and M is a homomorphic image of $V(\lambda)$. In particular, $M_\mu = 0$ unless $\mu \leq \lambda$.*

Proof. There is a natural isomorphism

$$\mathrm{Hom}_{Q_\sharp(n,d)}(U(\lambda), M\downarrow) \xrightarrow{\sim} \mathrm{Hom}_{Q(n,d)}(V(\lambda), M).$$

Choose an isomorphism $\theta : U(\lambda) \to M_\lambda \subseteq M$ of $Q_\sharp(n,d)$-supermodules and let $\theta\uparrow: V(\lambda) \to M$ be the corresponding $Q(n,d)$-supermodule homomorphism. This is non-zero, hence λ is admissible, and is surjective as M is generated by M_λ. This shows that M is a quotient of $V(\lambda)$, and the final statement about weights follows from Lemma 8.2. □

For admissible λ, define $L(\lambda)$ to be the head of $V(\lambda)$, i.e. $L(\lambda)$ is the largest completely reducible quotient supermodule of $V(\lambda)$. We remark that if $p = 0$ or $p > d$, then $Q(n,d)$ is semisimple by Lemma 4.1, so that $L(\lambda) = V(\lambda)$ in these cases.

8.4. **Lemma.** *The set $\{L(\lambda) \mid$ for all admissible $\lambda \in \Lambda(n,d)\}$ is a complete set of pairwise non-isomorphic irreducible $Q(n,d)$-supermodules. Moreover, $L(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

Proof. Let λ be admissible. We first claim that $V(\lambda)$ has a unique maximal subsupermodule, so that $L(\lambda)$ is irreducible. For let M, N be two maximal subsupermodules of $V(\lambda)$. Since $V(\lambda)_\lambda$ is irreducible over $Q_0(n,d)$ and generates $V(\lambda)$ over $Q(n,d)$, we must have that $M_\lambda = N_\lambda = 0$, so $(M+N)_\lambda = 0$. This shows that $M+N$ is a proper subsupermodule of $V(\lambda)$. Hence, $M = M+N = N$ by maximality, as required.

Evidently, for admissible $\lambda \neq \mu$, $L(\lambda)$ and $L(\mu)$ are not isomorphic, as they have different highest weights. Now suppose that L is an arbitrary irreducible $Q(n,d)$-supermodule. Choose λ maximal in the dominance order such that $L_\lambda \neq 0$. Then, by irreducibility, L must be a highest weight module of highest weight λ, so a quotient of $V(\lambda)$ by Lemma 8.3. Hence, $L \cong L(\lambda)$.

It remains to prove the statement about absolute irreducibility. First observe by adjointness that $\mathrm{Hom}_{Q(n,d)}(V(\lambda), L(\lambda)) \cong \mathrm{Hom}_{Q_\sharp(n,d)}(U(\lambda), L(\lambda)\downarrow) \cong \mathrm{End}_{Q_0(n,d)}(U(\lambda))$. Now there is a natural embedding $\mathrm{Hom}_{Q(n,d)}(L(\lambda), L(\lambda)) \hookrightarrow \mathrm{Hom}_{Q(n,d)}(V(\lambda), L(\lambda))$. To see that it is an isomorphism, observe that any $Q(n,d)$-homomorphism $V(\lambda) \to L(\lambda)$ annihilates the unique maximal submodule of $V(\lambda)$, hence induces a well-defined homomorphism $L(\lambda) \to L(\lambda)$. We have shown that $\mathrm{End}_{Q(n,d)}(L(\lambda)) \cong \mathrm{End}_{Q_0(n,d)}(U(\lambda))$. Now the final part of the lemma follows from Lemma 6.8 and Schur's lemma. □

9 Classification of admissible weights

We now proceed to give a combinatorial description of the admissible weights, to complete the classification of the irreducible $Q(n,d)$-supermodules. We make some definitions. Let $\Lambda^+(n,d)$ denote the set of all $\lambda \in \Lambda(n,d)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, i.e. λ is a partition of d with at most n non-zero parts. Let $\Lambda^+_p(n,d)$ denote the set of all $\lambda \in \Lambda^+(n,d)$ such that

$$\lambda_i = \lambda_{i+1} \quad \Rightarrow \quad p | \lambda_i \quad \text{for each } i = 1, 2, \ldots, n-1.$$

Call $\lambda \in \Lambda_p^+(n,d)$ *restricted* if either $p = 0$ or $p > 0$ and

$$\begin{cases} \lambda_i - \lambda_{i+1} \leq p & \text{if } p \nmid \lambda_i, \\ \lambda_i - \lambda_{i+1} < p & \text{if } p \mid \lambda_i \end{cases}$$

for each $i = 1, 2, \dots, n-1$. Let $\Lambda_p^+(n,d)_{\mathrm{res}}$ denote the set of all restricted $\lambda \in \Lambda_p^+(n,d)$.

We first construct another subsuperalgebra of $Q(n,d)$. Let $\mathscr{K}(n) = \bigoplus_{d \geq 0} \mathscr{K}(n,d)$ denote the $\mathbb{Z}$-graded superideal of $B(n)$ generated by the elements

$$\{b_{i,j} \mid i = 1, \dots, n, j = -1, \dots, -n\}.$$

It is a bisuperideal, so we can form the bisuperalgebra quotient

$$A(n) = B(n)/\mathscr{K}(n),$$

this being $\mathbb{Z}$-graded as $A(n) = \bigoplus_{d \geq 0} A(n,d)$ where $A(n,d) \cong B(n,d)/\mathscr{K}(n,d)$. For $i, j = 1, \dots, n$, set $c_{i,j} = b_{i,j} + \mathscr{K}(n)$. Observing that each $c_{i,j}$ has degree $\bar{0}$, $A(n) = A(n)_{\bar{0}}$ is precisely the free polynomial algebra on the generators $\{c_{i,j} \mid 1 \leq i, j \leq n\}$. So the dual superalgebra $S(n,d) = A(n,d)^*$ is just the usual classical Schur algebra as in [7] concentrated in degree $\bar{0}$. We identify $S(n,d)$ with the subsuperalgebra $\mathscr{K}(n,d)^\circ \subseteq Q(n,d)_{\bar{0}} \subseteq Q(n,d)$.

Now we treat the case $n = 2$, copying an argument due to Penkov [21, §7] in our setting.

9.1. **Lemma.** *Suppose that $n = 2$ and that $\lambda \in \Lambda(2,d)$ is an admissible weight. Then, either $\lambda_1 > \lambda_2$, or $\lambda_1 = \lambda_2 = c$ for some $c \geq 0$ with $p \mid c$.*

Proof. The restriction of $L(\lambda)$ to the ordinary Schur algebra $S(2,d) \subseteq Q(2,d)$ gives us an $S(2,d)$-module with maximal weight λ. We deduce from the classical theory that $\lambda_1 \geq \lambda_2$. To complete the proof, suppose for a contradiction that $\lambda_1 = \lambda_2 = c$ but that $p \nmid c$. So $d = 2c$. Now, there are no $\mu \in \Lambda^+(2,2c)$ with $\mu < \lambda$. Since we also know that $\dim L(\lambda)_\lambda = \dim U(\lambda) = 2$, we deduce by the classical representation theory of $S(2,2c)$ that $L(\lambda) \downarrow S(2,2c)$ splits as a direct sum of two irreducible $S(2,2c)$-modules both of highest weight λ. But such $S(2,2c)$-modules are one dimensional (being just a tensor power of the determinant module). This shows that $L(\lambda) = L(\lambda)_\lambda$, of dimension exactly two. Hence, $L(\lambda)_\nu = 0$ for all $\nu \neq \lambda$.

Define the following elements of $I(2,2c)$:

$$\begin{aligned}
\underline{i} &= (1, \dots, 1, -2; 2, \dots, 2, 2), & \underline{j} &= (1, \dots, 1, 2; 2, \dots, 2, 2), \\
\underline{k} &= (1, \dots, 1, 1; 2, \dots, 2, -1), & \underline{l} &= (1, \dots, 1, 1; 2, \dots, 2, 1), \\
\underline{s} &= (1, \dots, 1, -1; 2, \dots, 2, 2), & \underline{t} &= (1, \dots, 1, 1; 2, \dots, 2, -2), \\
\underline{u} &= (1, \dots, 1, -2; 2, \dots, 2, 1), & \underline{i}_\lambda &= (1, \dots, 1, 1; 2, \dots, 2, 2)
\end{aligned}$$

where the symbol ; is between the cth and $(c+1)$th entries. Now an explicit calculation using the product rule Theorem 4.5 shows that

$$\xi_{\underline{i}_\lambda, \underline{j}} \xi_{\underline{i}, \underline{i}_\lambda} = \xi_{\underline{s}, \underline{i}_\lambda} + \xi_{\underline{u}, \underline{i}_\lambda} \qquad \text{and} \qquad \xi_{\underline{i}_\lambda, \underline{k}} \xi_{\underline{l}, \underline{i}_\lambda} = \xi_{\underline{t}, \underline{i}_\lambda} + \xi_{\underline{u}, \underline{i}_\lambda}.$$

Hence,

$$\xi_{\underline{i}_\lambda,\underline{j}}\xi_{\underline{i},\underline{i}_\lambda} - \xi_{\underline{i}_\lambda,\underline{k}}\xi_{\underline{l},\underline{i}_\lambda} = \xi_{\underline{s},\underline{i}_\lambda} - \xi_{\underline{t},\underline{i}_\lambda}.$$

Using the previous paragraph and a weight argument, both terms on the left hand side of this equation act as zero on $L(\lambda)_\lambda$. Hence, the term $\xi_{\underline{s},\underline{i}_\lambda} - \xi_{\underline{t},\underline{i}_\lambda} \in \xi_\lambda Q_0(n,d)$ on the right hand side acts as zero on $L(\lambda)_\lambda \cong U(\lambda)$. But $\xi_\lambda Q_0(n,d) \cong C(2)$ according to (6.7), so as $U(2)$ is a faithful $C(2)$-supermodule, the non-zero element $\xi_{\underline{s},\underline{i}_\lambda} - \xi_{\underline{t},\underline{i}_\lambda}$ of $\xi_\lambda Q_0(n,d)$ cannot act as zero on $U(\lambda)$, a contradiction. □

Now observe that for $\lambda \in \Lambda(n,d)$, λ lies in $\Lambda_p^+(n,d)$ if and only if for each $i = 1, \dots, n-1$ $(\lambda_i, \lambda_{i+1})$ lies in $\Lambda_p^+(2, \lambda_i + \lambda_{i+1})$. So by an argument involving restriction to various quotients of $B(n)$ isomorphic to $B(2)$, we have the following corollary of Lemma 9.1:

9.2. **Corollary.** *If $\lambda \in \Lambda(n,d)$ is admissible, then $\lambda \in \Lambda_p^+(n,d)$.*

It remains to prove that every $\lambda \in \Lambda_p^+(n,d)$ is admissible, i.e. that there does exist some highest weight module of highest weight λ for each $\lambda \in \Lambda_p^+(n,d)$. We first give a construction of some highest weight modules in the case $p > 0$ using a Frobenius twist argument. Recall from earlier in the section that $A(n)$ denotes the free polynomial algebra on generators $\{c_{i,j} \mid 1 \leq i,j \leq n\}$, viewed as a bialgebra as in the classical polynomial representation theory of $GL(n)$ [7]. In particular, we can view $A(n)$ is a bisuperalgebra concentrated in degree $\bar{0}$.

9.3. **Lemma.** *If $p > 0$, the unique algebra map $\sigma : A(n) \to B(n)$, such that $c_{i,j} \mapsto b_{i,j}^p$ for all $1 \leq i,j \leq n$, is a bisuperalgebra embedding.*

Proof. This is a routine check of relations, similar to that carried out in [6, §2.3]. □

In view of the lemma, there is a natural restriction functor

$$\mathrm{Fr} : \mathbf{mod}(A(n)) \to \mathbf{mod}(B(n)).$$

On objects, Fr is defined by sending an $A(n)$-cosupermodule M with structure map $\eta : M \to M \otimes A(n)$ to the $B(n)$-cosupermodule equal to M as a superspace with structure map $(\mathrm{id} \otimes \sigma) \circ \eta$; we call $\mathrm{Fr}\, M$ the *Frobenius twist* of M. On morphisms, Fr sends a morphism to the same linear map but regarded instead as a $B(n)$-cosupermodule map. We note that if M is a polynomial $A(n)$-cosupermodule of degree d, then $\mathrm{Fr}\, M$ is a $B(n, pd)$-cosupermodule. Also, let $\mathrm{Fr} : X(n) \to X(n)$ be the linear map determined by $\mathrm{Fr}(x^\lambda) = x^{p\lambda}$ for each $\lambda \in \Lambda(n)$, where $p\lambda$ denotes $(p\lambda_1, \dots, p\lambda_n)$. Then, the formula

$$\mathrm{ch}(\mathrm{Fr}\, M) = \mathrm{Fr}(\mathrm{ch}\, M)$$

describes the effect of the functor Fr at the level of characters.

9.4. **Lemma.** *Suppose that $\lambda \in \Lambda(n, d_1)$ is an admissible weight, and that $\mu \in \Lambda^+(n, d_2)$ is arbitrary. Then, $\lambda + p\mu \in \Lambda(n, d_1 + pd_2)$ is an admissible weight. Moreover, all non-zero weights of $L(\lambda + p\mu)$ are of the form $\lambda' + p\mu'$ for $\lambda' \leq \lambda$ and $\mu' \leq \mu$.*

Proof. If $p = 0$, there is nothing to prove. Otherwise, by the classical theory, there exists an irreducible $A(n)$-comodule $L'(\mu)$ of highest weight μ. Regard $L'(\mu)$ instead as an $A(n)$-cosupermodule concentrated in degree $\bar{0}$ (say) and consider the $B(n)$-cosupermodule

$$M = L(\lambda) \otimes \mathrm{Fr}\, L'(\mu).$$

It is a $B(n, d_1 + pd_2)$-cosupermodule, hence a $Q(n, d_1 + pd_2)$-supermodule. Its non-zero weights are of the form $\lambda' + p\mu'$ for $\lambda \leq \lambda$ and $\mu' \leq \mu$, and the weight $\lambda + p\mu$ definitely appears as a weight of M. Hence, there exists a highest weight module of highest weight $\lambda + p\mu$, so $\lambda + p\mu$ is admissible. The statement about weights follows because $L(\lambda + p\mu)$ must then be a subquotient of M. $\square$

Now we are in a position to complete the classification of admissible weights by a counting argument. Recall the definition of the idempotent ξ_ω from §6.

9.5. **Theorem.** (i) *$\lambda \in \Lambda(n, d)$ is admissible if and only if $\lambda \in \Lambda_p^+(n, d)$.*

(ii) *Assuming that $n \geq d$ and $\lambda \in \Lambda_p^+(n, d)$, we have that $\xi_\omega L(\lambda) \neq 0$ if and only if $\lambda \in \Lambda_p^+(n, d)_{\mathrm{res}}$.*

Proof. Recalling Corollary 9.2, we just need to show for (i) that if $\lambda \in \Lambda_p^+(n, d)$, then λ is admissible. We consider first the case $n \geq d$, and proceed by induction on $d = 0, 1, \dots, n$. The result is trivially true in case $d = 0$. For $n \geq d > 0$, take $\lambda \in \Lambda_p^+(n, d)$. Suppose first that $\lambda \notin \Lambda_p^+(n, d)_{\mathrm{res}}$. Then, we can write $\lambda = \lambda_1 + p\lambda_2$ where $\lambda_1 \in \Lambda_p^+(n, d_1)$ and $\lambda_2 \in \Lambda^+(n, d_2)$ for some d_1, d_2 with $d = d_1 + pd_2$ and $d_2 \neq 0$. By induction, λ_1 is admissible, so we deduce from Lemma 9.4 that λ is admissible, and moreover that $\xi_\omega L(\lambda) = 0$. But by Lemma 6.3, there are exactly $|\mathscr{RP}_p(d)| = |\Lambda_p^+(n, d)_{\mathrm{res}}|$ non-isomorphic irreducible $Q(n, d)$-supermodules not annihilated by ξ_ω. In view of Corollary 9.2, this means that all $\lambda \in \Lambda_p^+(n, d)_{\mathrm{res}}$ must both be admissible and satisfy $\xi_\omega L(\lambda) \neq 0$, else we end up with too few such modules.

Now suppose that $n < d$ and choose $m \geq d$. Let $e \in Q(m, d)$ be the idempotent defined in (6.4), and also recall the embedding $\Lambda(n, d) \hookrightarrow \Lambda(m, d)$ there. Take $\lambda \in \Lambda_p^+(n, d)$. Then, viewing λ as an element of $\Lambda_p^+(m, d)$, we have already shown in the previous paragraph that λ is admissible for $Q(m, d)$, so that there exists an irreducible $Q(m, d)$-supermodule $L(\lambda)$ of highest weight λ. Clearly, $eL(\lambda)_\lambda \neq 0$ as $\lambda \in \Lambda(n, d)$. Taking into account Lemma 6.5 and Corollary 2.13, $eL(\lambda)$ is an irreducible $Q(n, d)$-supermodule of highest weight λ. $\square$

10 Decomposition numbers

In Theorem 9.5(i) and Lemma 8.4, we have classified the irreducible $Q(n, d)$-supermodules; they are precisely the supermodules $\{L(\lambda) \mid \lambda \in \Lambda_p^+(n, d)\}$. Applying Lemma 5.1, we have equivalently determined the irreducible $B(n)$-cosupermodules. Let $\Lambda_p^+(n) = \bigcup_{d \geq 0} \Lambda_p^+(n, d)$ denote the set of all p-strict partitions with at most n non-zero parts. Then, we have shown:

10.1. **Theorem.** *The $B(n)$-cosupermodules $\{L(\lambda) \mid \lambda \in \Lambda_p^+(n)\}$ give a complete set of pairwise non-isomorphic irreducible $B(n)$-cosupermodules. Moreover, $L(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

Next we turn our attention to constructing the irreducible representations of the Sergeev superalgebra $W(d)$. Let $n \geq d$, and identify $\Lambda_p^+(n,d)$ with the set $\mathscr{P}_p(d)$ of all p-strict partitions of d. Then, $\Lambda_p^+(n,d)_{\mathrm{res}}$ is identified with $\mathscr{RP}_p(d) \subseteq \mathscr{P}_p(d)$. Also let $\xi_\omega \in Q(n,d)$ be the idempotent from §6. For $\lambda \in \mathscr{RP}_p(d)$, define the $W(d)$-supermodule

$$M(\lambda) := \xi_\omega L(\lambda).$$

We should note that this definition is independent of the particular choice of $n \geq d$ up to natural isomorphism (this is proved in a standard way, see e.g. [5, §3.5]). The following result is immediate from Theorem 9.5(ii) and Corollary 2.13:

10.2. **Theorem.** *The modules $\{M(\lambda) \mid \lambda \in \mathscr{RP}_p(d)\}$ give a complete set of pairwise non-isomorphic irreducible $W(d)$-supermodules. Moreover, $M(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

In order to obtain a labelling for *all* irreducible $W(d)$-modules, not just supermodules, we know by Lemma 2.3 that if $M(\lambda)$ is self-associate, it decomposes as $M(\lambda,+) \oplus M(\lambda,-)$ for two non-isomorphic irreducible $W(d)$-modules $M(\lambda,+), M(\lambda,-)$. By Corollary 2.8, the modules

$$\{M(\lambda) \mid \lambda \in \mathscr{RP}_p(d), h_{p'}(\lambda) \text{ even}\} \cup \{M(\lambda,+), M(\lambda,-) \mid \lambda \in \mathscr{RP}_p(d), h_{p'}(\lambda) \text{ odd}\}$$

then give a complete set of pairwise non-isomorphic irreducible $W(d)$-modules.

To pass to the projective representations of the symmetric group, we use the functors F and G from §3 together with Corollary 3.5. Suppose first that d is even. For $\lambda \in \mathscr{RP}_p(d)$, set $D(\lambda) = GM(\lambda)$, an irreducible $S(d)$-supermodule which is absolutely irreducible if and only if $M(\lambda)$ is absolutely irreducible, which is if and only if $h_{p'}(\lambda)$ is even. In the case that d is odd, take $\lambda \in \mathscr{RP}_p(d)$. If $h_{p'}(\lambda)$ is even, we set $D(\lambda) = GM(\lambda)$ as before, giving us a self-associate irreducible $S(d)$-supermodule. If $h_{p'}(\lambda)$ is odd, there is an absolutely irreducible $S(d)$-supermodule $D(\lambda)$, unique up to isomorphism, such that $M(\lambda) \cong FD(\lambda)$. Then, recalling Corollary 3.5, we have:

10.3. **Theorem.** *The modules $\{D(\lambda) \mid \lambda \in \mathscr{RP}_p(d)\}$ give a complete set of pairwise non-isomorphic irreducible $S(d)$-supermodules. Moreover, $D(\lambda)$ is absolutely irreducible if and only if $d - h_{p'}(\lambda)$ is even.*

If $\lambda \in \mathscr{RP}_p(d)$ and $d - h_{p'}(\lambda)$ is odd, we can decompose $D(\lambda) \cong D(\lambda,+) \oplus D(\lambda,-)$ as a direct sum of two non-isomorphic irreducible $S(d)$-modules, and by Corollary 2.8 the modules

$$\{D(\lambda) \mid \lambda \in \mathscr{RP}_p(d), d - h_{p'}(\lambda) \text{ even}\} \cup \{D(\lambda,+), D(\lambda,-) \mid \lambda \in \mathscr{RP}_p(d), d - h_{p'}(\lambda) \text{ odd}\}$$

then give a complete set of pairwise non-isomorphic irreducible $S(d)$-modules. We have thus determined the irreducible projective representations of S_d over $\Bbbk$.

The next theorem explains how to obtain the irreducible projective representations of the alternating group A_d from these. Let $A(d) = S(d)_{\bar{0}}$. Providing $d > 7$, this is up to isomorphism the only twisted group algebra of A_d over $\Bbbk$, other than the group algebra $\Bbbk A_d$ itself. The following theorem is proved by arguments analogous to the Clifford theory for groups with normal subgroups of index two.

10.4. **Theorem.** *Let $\lambda \in \mathscr{RP}_p(d)$. If $d - h_{p'}(\lambda)$ is even, $D(\lambda)\downarrow_{A(d)} \cong E(\lambda,+) \oplus E(\lambda,-)$ for two non-isomorphic irreducible $A(d)$-modules $E(\lambda,+), E(\lambda,-)$. If $d - h_{p'}(\lambda)$ is odd, $D(\lambda)\downarrow_{A(d)} \cong E(\lambda) \oplus E(\lambda)$ for a single irreducible $A(d)$-module $E(\lambda)$. The modules*

$$\{E(\lambda) \mid \lambda \in \mathscr{RP}_p(d), d - h_{p'}(\lambda) \text{ odd}\} \cup \{E(\lambda,+), E(\lambda,-) \mid \lambda \in \mathscr{RP}_p(d), d - h_{p'}(\lambda) \text{ even}\}$$

then give a complete set of pairwise non-isomorphic irreducible $A(d)$-modules.

10.5. **Remark.** We have assumed up to now that $\Bbbk$ is algebraically closed. In fact, the construction of the irreducible (super)modules of $Q(n,d)$, $W(d)$, $S(d)$ and $A(d)$ that we have described can be carried out in precisely the same way over any field $\Bbbk$ of characteristic different from 2 providing only that $\Bbbk$ contains square roots of all $\pm 1, \dots, \pm d$. In fact, any such field is a *splitting field* for each of the algebras $Q(n,d), W(d)$, $S(d)$ and $A(d)$. This is proved by reducing using a Schur functor argument to the case of $Q(n,d)$, where as explained in the proof of Lemma 8.4,

$$\mathrm{End}_{Q(n,d)}(L(\lambda)) \cong \mathrm{End}_{Q_0(n,d)}(U(\lambda)).$$

If $\Bbbk$ contains square roots of all $\pm 1, \dots, \pm d$, then $\Bbbk$ is a splitting field for each of the Clifford superalgebras $C(1), \dots, C(d)$, hence for $Q_0(n,d)$. So the right hand side is then one or two dimensional according to whether $L(\lambda)$ is absolutely irreducible or self-associate, as required to prove that $\Bbbk$ is a splitting field.

We conclude with some discussion of decomposition numbers. It is immediate from highest weight theory that the character map $\mathrm{ch} : \mathrm{Grot}(B(n)) \to X(n)$ described at the end of §6 is an embedding of the Grothendieck ring of the category of $B(n)$-cosupermodules into $X(n)$. Set $L_\lambda = \mathrm{ch}\, L(\lambda)$, for $\lambda \in \Lambda_p^+(n)$. Then, the elements

$$\{L_\lambda \mid \lambda \in \Lambda_p^+(n)\}$$

of $X(n)$ form a $\mathbb{Z}$-basis for the image of ch. For $\lambda \in \Lambda^+(n)$, *Schur's P-function* P_λ is defined by:

$$P_\lambda = \sum_{w \in S_n / S_\lambda} w \left\{ x^\lambda \frac{\prod_{\lambda_i > \lambda_j} (x_i + x_j)}{\prod_{\lambda_i > \lambda_j} (x_i - x_j)} \right\}, \tag{10.6}$$

where S_λ denotes the stablizier of x^λ in S_n and S_n / S_λ is some choice of left coset representatives. This is the definition from [16, III(2.2)] (with t there equal to -1, compare [16, III.8]). For $\lambda \in \Lambda_p^+(n)$, let

$$E_\lambda = 2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor} P_\lambda.$$

The E_λ arise naturally as certain Euler characteristics, in an analogous way to the construction in the work of Penkov and Serganova in characteristic 0, see [22, Prop.1] and [23].

(Fuller details in the positive characteristic case will appear elsewhere.) In particular, E_λ is an alternating sum of characters of $B(n)$-cosupermodules. Since E_λ and L_λ have the same leading term $2^{\lfloor (h_{p'}(\lambda)+1)/2\rfloor} x^\lambda$ plus a linear combination of lower terms lower with respect to the dominance order, it follows easily that

$$\{E_\lambda \mid \lambda \in \Lambda_p^+(n)\}$$

also forms a $\mathbb{Z}$-basis for the image of ch. So we can write

$$E_\lambda = \sum_{\mu \in \Lambda_p^+(n)} d_{\lambda,\mu} L_\mu$$

for uniquely determined $d_{\lambda,\mu} \in \mathbb{Z}$ with $d_{\lambda,\lambda} = 1$ and $d_{\lambda,\mu} = 0$ if $\mu \not\leq \lambda$. We will call the matrix $D = (d_{\lambda,\mu})_{\lambda,\mu \in \Lambda_p^+(n,d)}$ the *decomposition matrix* of $Q(n,d)$ in characteristic p.

Now suppose that $(\Bbbk, R, K)$ is a p-modular system with K sufficiently large (specifically, containing square roots of $\pm 1, \dots, \pm d$). So, R is a complete discrete valuation ring, K is its field of fractions of characteristic 0 and our fixed algebraically closed field $\Bbbk$ of characteristic p is its residue field. The bisuperalgebra $B(n)$ can be defined in exactly the same as in §5 but over the ground ring R, giving us an R-free R-bisuperalgebra $B(n)_R$ such that $B(n) \cong B(n)_R \otimes_R \Bbbk$. Set $Q(n,d)_R = \mathrm{Hom}_R(B(n,d)_R, R)$ to obtain an R-form of the Schur superalgebra $Q(n,d)$. So, $Q(n,d)_R$ is R-free as an R-module and $Q(n,d) \cong Q(n,d)_R \otimes_R \Bbbk$; we will from now on identify the two. Also, set $Q(n,d)_K = Q(n,d)_R \otimes_R K$, the analogous Schur superalgebra over the ground field K. Similarly, we can define an R-form $Q_0(n,d)_R$ of $Q_0(n,d)$, and set $Q_0(n,d)_K = Q_0(n,d)_R \otimes_R K$. We will view $Q(n,d)_R$ and $Q_0(n,d)_R$ as R-subsuperalgebras of $Q(n,d)_K$.

For $\lambda \in \Lambda_0^+(n,d)$, let $V(\lambda)_K$ denote the irreducible $Q(n,d)_K$-supermodule of highest weight λ, constructed as in (8.1). By Sergeev's character formula [25, Theorem 4],

$$\mathrm{ch}\, V(\lambda)_K = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} P_\lambda$$

where $h(\lambda)$ is the number of non-zero parts of λ. Denote the highest weight space of $V(\lambda)_K$ by $U(\lambda)_K$; this is precisely the $Q_0(n,d)_K$-supermodule defined as in §6. Now, the construction of $U(\lambda)_K$ can be carried out over R instead, because R contains square roots of each $\pm\lambda_i$, giving us a finitely generated R-free $Q_0(n,d)_R$-subsupermodule $U(\lambda)_R$ of $U(\lambda)_K$ such that $U(\lambda)_K \cong U(\lambda)_R \otimes_R K$. Let $V(\lambda)_R$ denote the $Q(n,d)_R$-subsupermodule of $V(\lambda)_K$ generated by $U(\lambda)_R$. Then, $V(\lambda)_R$ is a finitely generated R-free R-module such that $V(\lambda)_K \cong V(\lambda)_R \otimes_R K$. Now set $\overline{V}(\lambda) := V(\lambda)_R \otimes_R \Bbbk$. This gives us a $Q(n,d)$-supermodule such that

$$\mathrm{ch}\, \overline{V}(\lambda) = \mathrm{ch}\, V(\lambda)_K = 2^{\lfloor (h(\lambda)+1)/2 \rfloor - \lfloor (h_{p'}(\lambda)+1)/2 \rfloor} E_\lambda.$$

In particular, we deduce:

10.7. **Theorem.** *For $\lambda \in \Lambda_0^+(n)$ and $\mu \in \Lambda_p^+(n)$, the decomposition number $d_{\lambda,\mu}$ defined above is a non-negative integer.*

One can hope that in fact $d_{\lambda,\mu} \geq 0$ *for all* $\lambda, \mu \in \Lambda_p^+(n)$.

Finally, we relate the decomposition matrix D of $Q(n,d)$ for $n \geq d$ to the decomposition matrices of the superalgebras $W(d)$ and $S(d)$. Using the subscript K to indicate that we are working over the ground field K instead of our usual $\Bbbk$, we have irreducible $W(d)_K$- (resp. $S(d)_K$-) supermodules labelled by strict partitions $\lambda \in \mathscr{P}_0(d)$, which we denote by $M(\lambda)_K$ and $D(\lambda)_K$ respectively. By a straightforward extension of Brauer's theory, we can reduce these modulo p to obtain $W(d)$- (resp. $S(d)$-) supermodules $\overline{M}(\lambda)$ and $\overline{D}(\lambda)$. These are not uniquely determined up to isomorphism, but at least the multiplicities of composition factors are unique. So we obtain well-defined *decomposition matrices* $D^S = (d^S_{\lambda,\mu})$ and $D^W = (d^W_{\lambda,\mu})$ of $S(d)$ and $W(d)$ respectively, for $\lambda \in \mathscr{P}_0(d), \mu \in \mathscr{RP}_p(d)$, determined by the equations

$$[\overline{M}(\lambda)] = \sum_{\mu \in \mathscr{RP}_p(d)} d^W_{\lambda,\mu}[M(\mu)], \qquad [\overline{D}(\lambda)] = \sum_{\mu \in \mathscr{RP}_p(d)} d^S_{\lambda,\mu}[D(\mu)]$$

written in the Grothendieck groups of $\mathbf{mod}(W(d))$ and $\mathbf{mod}(S(d))$ respectively. The final theorem relates these decomposition numbers to those of the Schur superalgebra $Q(n,d)$:

10.8. **Theorem.** *Let $D = (d_{\lambda,\mu})_{\lambda,\mu\in\Lambda^+_p(n,d)}$ be the decomposition matrix of $Q(d,d)$ in characteristic p, as defined above. Then, for any $\lambda \in \mathscr{P}_0(d)$ and $\mu \in \mathscr{RP}_p(d)$,*

$$d^W_{\lambda,\mu} = 2^{\lfloor (h(\lambda)+1)/2\rfloor - \lfloor (h_{p'}(\lambda)+1)/2\rfloor} d_{\lambda,\mu}.$$

Moreover, if d is even,

$$d^S_{\lambda,\mu} = d^W_{\lambda,\mu},$$

while if d is odd,

$$d^S_{\lambda,\mu} = \begin{cases} d^W_{\lambda,\mu} & \text{if } h(\lambda) - h_{p'}(\mu) \text{ is even,} \\ 2d^W_{\lambda,\mu} & \text{if } h(\lambda) \text{ is even and } h_{p'}(\mu) \text{ is odd,} \\ \frac{1}{2}d^W_{\lambda,\mu} & \text{if } h(\lambda) \text{ is odd and } h_{p'}(\mu) \text{ is even.} \end{cases}$$

Proof. The Schur functor coming from the idempotent ξ_ω can be defined over the ground ring R, using an R-integral version of Theorem 6.2. Using that Schur functors commute with base change, one sees that $[\xi_\omega \overline{V}(\lambda)] = [\overline{M}(\lambda)]$ (equality written in the Grothendieck group). In particular, it follows from this by exactness of Schur functors that $d^W_{\lambda,\mu} = d_{\lambda,\mu}$.

Similarly, the functors F from §3 can be defined over the ground ring R, and F evidently commutes with base change. So in the case that d is even, $[F\overline{D}(\lambda)] = [\overline{M}(\lambda)]$ and $FD(\mu) = M(\mu)$ by Theorem 3.4 over K or $\Bbbk$ respectively, hence $d^S_{\lambda,\mu} = d^W_{\lambda,\mu}$.

Finally, suppose that d is odd. Applying Theorem 3.4 and Lemma 2.9 over K or $\Bbbk$ respectively, we have that

$$[F\overline{D}(\lambda)] = \begin{cases} [\overline{M}(\lambda)] & \text{if } h(\lambda) \text{ is odd,} \\ 2[\overline{M}(\lambda)] & \text{if } h(\lambda) \text{ is even;} \end{cases}$$

$$[FD(\mu)] = \begin{cases} [M(\mu)] & \text{if } h_{p'}(\mu) \text{ is odd,} \\ 2[M(\mu)] & \text{if } h_{p'}(\lambda) \text{ is even.} \end{cases}$$

The theorem follows from these equations together with exactness of F. $\square$

Thus our results show that the decomposition matrices for projective representations of the symmetric group S_d can be deduced from knowledge of the decomposition matrix of the Schur superalgebra $Q(d,d)$. In [14], a precise conjecture is made relating decomposition matrices for projective representations of S_d to the specialization at $q = 1$ of certain polynomials $d_{\lambda,\mu}(q)$ arising as coefficients of the canonical basis of the identity component of the Fock space of $U_q(A^{(2)}_{p-1})$. Indeed, it appears that for $\lambda \in \mathscr{P}_p(d), \mu \in \mathscr{R}\mathscr{P}_p(d)$, the integer $d_{\lambda,\mu}(1)$ as defined in [14] should equal the decomposition number $d_{\lambda,\mu}$ of $Q(d,d)$ (as defined above) providing $d < p^2$. This statement is essentially a reformulation of the conjecture made by Leclerc and Thibon in [14]. It would be interesting to extend the Leclerc-Thibon construction of the canonical basis of the identity component of the Fock space of $U_q(A^{(2)}_{p-1})$ to the entire Fock space, as was done in [13] for the case of $U_q(A^{(1)}_{p-1})$, to obtain a conjectural algorithm for computing $d_{\lambda,\mu}$ for *all* $\mu \in \mathscr{P}_p(d)$.

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