ON COMPARING THE COHOMOLOGY OF GENERAL LINEAR AND SYMMETRIC GROUPS

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1 Introduction

1.1 Let k be an algebraically closed field, $\mathsf{GL}_n(k)$ be the general linear group over k, and Σ_d be the symmetric group on d letters. For $k = \mathbb{C}$, Frobenius and Schur discovered that the commuting actions of $\mathsf{GL}_n(k)$ and Σ_d on $V^{\otimes d}$ can be used to relate the character theory of these two groups. For modular representations of these groups the relationship between their representation theories is not as direct. However, James [12] showed that the decomposition matrix of the symmetric group is a submatrix of the decomposition matrix of the general linear group, and Erdmann [9] used tilting modules to prove that one can also recover the decomposition numbers of the general linear groups from those of the symmetric groups.

Let $M(n,d)$ be the category of polynomial representations of $\mathsf{GL}_n(k)$ of a fixed degree $d \leq n$. The Schur functor $\mathcal{F}$ is a certain covariant exact functor from $M(n,d)$ to modules for the group algebra $k\Sigma_d$ ([10]). Some of the aforementioned results can be proved by using $\mathcal{F}$. So it is natural to hope that the Schur functor can also be used to compare the cohomogy theories of $\mathsf{GL}_n(k)$ and Σ_d. In this paper we address the following question. Let $M, N \in M(n,d)$. When is it true that

$$\operatorname{Ext}^i_{\mathsf{GL}_n(k)}(M,N) \cong \operatorname{Ext}^i_{k\Sigma_d}(\mathcal{F}(M), \mathcal{F}(N))\,? \tag{1.1.1}$$

Here $\operatorname{Ext}^i_{\mathsf{GL}_n(k)}$ is taken in the category $M(n,d)$. By [5, 2.1f], this is equivalent to considering extensions in the category of all rational $\mathsf{GL}_n(k)$-modules. The case where $i = 1$ and M, N are irreducible representations is of particular importance.

The precise relationship between the two cohomology groups is given by the spectral sequence of [7]. It starts with extensions for $\mathsf{GL}_n(k)$ and converges to extensions for $k\Sigma_d$. However, to use the spectral sequence one needs to compute the higher right derived functors $R^\bullet\mathcal{G}(-) \cong \operatorname{Ext}^\bullet_{k\Sigma_d}(V^{\otimes d}, -)$, where $\mathcal{G}$ is a right adjoint functor to $\mathcal{F}$ (see Section 2.2). These derived functors are used extensively throughout the paper.

1.2 To describe the results of the paper we need some notation. Let us denote the simple (polynomial) $\mathsf{GL}_n(k)$ module with highest weight λ by $L(\lambda)$. We also write $\Delta(\lambda)$ and $\nabla(\lambda)$ for the standard and costandard $\mathsf{GL}_n(k)$-modules with highest weight λ, respectively. It is well-known that $\mathcal{F}(L(\lambda))$ is non-zero if and only if λ is p-restricted, and

$$\{\mathcal{F}(L(\lambda)) \mid L(\lambda) \in M(n,d), \text{ and } \lambda \text{ is } p\text{-restricted}\}$$

is a complete set of the simple $k\Sigma_d$-modules up to isomorphism. If λ is p-restricted, we denote $\mathcal{F}(L(\lambda))$ by D_λ.

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A simple $k\Sigma_d$-module is called *completely splittable* (CS for short) if its restriction to any Young subgroup is semisimple. These modules were introduced in [15], see also [20, 17, 19]. It is possible to say explicitly which of the D_λ are CS, see [15] and Section 4.2.

Now we describe the contents of the paper in greater detail. In Section 2, the basic facts about the Schur functor are reviewed. We also define the spectral sequence constructed in [7] and state the elementary properties of the higher right derived functors $R^\bullet\mathcal{G}(-)$. In the following section the image of $\mathcal{G}$ on twisted Specht and twisted Young modules is computed (for $p>3$). The main result of the section (Theorem 3.2) can be interpreted as the isomorphism (1.1.1) for important classes of modules. Section 4 deals with CS modules. Our result in Theorem 4.4(a) says that

$$\mathcal{G}(D_\lambda) = L(\lambda). \tag{1.2.1}$$

if D_λ is CS and non-trivial (i.e. $D_\lambda \ncong k$). We note that the image of $\mathcal{G}$ on the simple $k\Sigma_d$-modules is generally very complicated. According to [7, 3.3], $\mathcal{G}(D_\lambda)$ can be described as the largest submodule of the injective hull of $L(\lambda)$ in the category $M(n,d)$ whose only p-restricted composition factor is $L(\lambda)$. Informally, $\mathcal{G}(D_\lambda)$ is $L(\lambda)$ with as many non-p-restricted modules on top as possible. In the light of this description the property (1.2.1) is quite remarkable. Using this result we show in Theorem 4.4(b) that if λ and μ are p-restricted, D_λ is CS, and $D_\lambda \ncong k$ then

$$\operatorname{Ext}^1_{\mathsf{GL}_n(k)}(L(\mu), L(\lambda)) \cong \operatorname{Ext}^1_{k\Sigma_d}(D_\mu, D_\lambda). \tag{1.2.2}$$

This again provides us with many cases where (1.1.1) is indeed an isomorphism.

Section 5 contains vanishing results about the cohomology of symmetric groups with coefficients in CS modules. This information is later used to prove the vanishing of some $R^\bullet\mathcal{G}(D_\lambda)$. The results of Section 5 allow us to generalize the isomorphism (1.2.2) to higher Ext-groups in a suitable range of degrees. This result is a portion of two general theorems on stability of extensions given in Sections 6.1 and 6.3, which give further examples where (1.1.1) holds. For example, Corollary 6.1a claims that

$$\mathrm{H}^i(\Sigma_d, \mathcal{F}(N)\otimes \mathrm{sgn}) \cong \operatorname{Ext}^i_{\mathsf{GL}_d(k)}(\delta, N) \quad \text{for } 0 \le i \le p-3.$$

Here, sgn is the one-dimensional *sign representation* for Σ_d, δ is the one-dimensional *determinant representation* for $\mathsf{GL}_d(k)$, and $N \in M(n,d)$. Note that Cline, Parshall and Scott [4, (12.4)] have a similar result in *non-describing characteristic* relating the cohomology of the finite general linear group with cohomology for the q-Schur algebra. Their approach relies on using the Deodhar complex and is quite different from ours.

One can improve the stability results on the symmetric group cohomology with coefficients in a simple module if one considers the following setting. Let V be the natural n-dimensional $\mathsf{GL}_n(k)$-module and $S^d(V)$ be the d-th symmetric power of V. Corollary 6.3(b) shows that for $n \ge d$ and $0 \le i \le 2(p-2)$,

$$\mathrm{H}^i(\Sigma_d, D_\mu) \cong \operatorname{Ext}^i_{\mathsf{GL}_n(k)}(S^d(V), L(\mu)).$$

To interpret this statement in the context of (1.1.1) one needs to note that $\mathcal{F}(S^d(V)) \cong k$.

Theorem 6.1 can also be used to prove a conjecture on the cohomology of dual Specht modules made in [2]. In fact, we get an even stronger result (Corollary 6.2) that for any Specht module S^λ over $k\Sigma_d$ and $1 \le i \le p-3$,

$$\mathrm{H}^i(\Sigma_d, (S^\lambda)^*) = 0. \tag{1.2.3}$$

This result can be interpreted as an isomorphism in (1.1.1) with $M = \wedge^d(V)$ and N being any costandard module. The vanishing result (1.2.3) along with the work in Section 3 is used in Section 6.4 to prove stability results for extensions of Specht modules. This is followed by similar results in Section 6.5 on extensions of Young modules. Finally, in Section 6.6 we obtain some results on cohomology of the alternating groups.

2 Comparing $\mathsf{GL}_n(k)$ and $k\Sigma_d$

2.1 The Schur functor. The basic references for this section are [10], [11]. Let k be an algebraically closed field of characteristic $p > 0$ and let $G = \mathsf{GL}_n(k)$. The Schur algebra $S(n,d)$ is the finite-dimensional associative k-algebra $\mathrm{End}_{k\Sigma_d}(V^{\otimes d})$ where V is the natural representation of G. The category $M(n,d)$ of polynomial G-modules of a fixed degree $d \geq 0$ is equivalent to the category of modules for $S(n,d)$ and we do not distinguish between the two categories from now on. We denote by $\mathrm{Mod}(k\Sigma_d)$ (resp. $\mathrm{mod}(k\Sigma_d)$) the category of all (resp. all finite dimensional) $k\Sigma_d$-modules.

Throughout the paper we assume that $n \geq d$. Let $e = \zeta_{(1,2,\dots,d)(1,2,\dots,d)}$ be the idempotent in $S(n,d)$ described in [10, (6.1)]. Then $eS(n,d)e \cong k\Sigma_d$. The Schur functor $\mathcal{F}$ is the covariant exact functor from $M(n,d)$ to $\mathrm{Mod}(k\Sigma_d)$ defined on objects by $\mathcal{F}(M) = eM$.

The simple $S(n,d)$-modules are in bijective correspondence with set of partitions of d. We will denote this set by $\Lambda = \Lambda^+(n,d)$ and the corresponding simple $S(n,d)$-module by $L(\lambda)$ for $\lambda \in \Lambda$. Note that one can also identify Λ as the set of dominant polynomial weights of G of degree d. Moreover, if $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover of $L(\lambda)$ and $T(\lambda)$ be the corresponding tilting module. There exists a duality on $M(n,d)$ fixing simple modules called the transpose dual. This duality will be denoted by τ. The duality τ and the usual duality '$*$' in $\mathrm{mod}(k\Sigma_d)$ are compatible in the sense that $eM^\tau \cong (eM)^*$ for any finite dimensional $M \in M(n,d)$.

A partition $(\lambda_1, \lambda_2, \dots)$ is called *p-restricted* if $\lambda_i - \lambda_{i+1} \leq p-1$ for all i. As mentioned in Section 1.2, we label the simple $k\Sigma_d$-modules by the p-restricted partitions $\lambda \in \Lambda$. The set of the p-restricted partitions of d will be denoted by Λ_{res}. A partition λ is called *p-regular* if its *transpose* λ' is p-restricted. We denote the set of all p-regular partitions of d by Λ_{reg}. In [11], the simple $k\Sigma_d$-modules are labelled by the p-regular partitions and denoted by D^λ. We will use both parametrizations so note a result from [10, § 6]:

$$D^\lambda \cong D_{\lambda'} \otimes \mathrm{sgn} \quad \text{for any } \lambda \in \Lambda_{\mathrm{reg}}. \tag{2.1.1}$$

The Specht, Young, and permutation modules over $k\Sigma_d$ corresponding to a partition $\lambda \in \Lambda$ are denoted by S^λ, Y^λ, and M^λ, respectively. In particular M^λ is the module induced from the trivial module over the Young subgroup Σ_λ. One has following correspondences between $S(n,d)$-modules and $k\Sigma_d$-modules under $\mathcal{F}$ (see [10, § 6] and [6, 3.5, 3.6]):

$$\mathcal{F}(\nabla(\lambda)) = S^\lambda, \quad \mathcal{F}(\Delta(\lambda)) = (S^\lambda)^*, \quad \mathcal{F}(P(\lambda)) = Y^\lambda, \quad \mathcal{F}(T(\lambda)) = Y^{\lambda'} \otimes \mathrm{sgn}.$$

2.2 Spectral sequences. Let $A = S(n,d)$ and $eAe = k\Sigma_d$ with $n \geq d$. The Schur functor $\mathcal{F}$ can be represented as a Hom functor and a tensor functor: $\mathcal{F}(M) \cong eM \cong \mathrm{Hom}_A(Ae, M) \cong eA \otimes_A M$. By using this identification, $\mathcal{F}$ admits a right adjoint functor, a right adjoint $\mathcal{G}$ defined by

$$\mathcal{G}(N) = \mathrm{Hom}_{eAe}(eA, N).$$

Furthermore, $\mathcal{G}$ is a left inverse to $\mathcal{F}$. Since $Ae \cong V^{\otimes d}$, and $\mathcal{G}$ takes injective eAe-modules to injective A-modules, one can construct a spectral sequence [7, 2.2]:

Theorem (A). *Let $M \in M(n,d)$, $N \in \mathrm{Mod}(k\Sigma_d)$ with $n \geq d$. There exists a first-quadrant Grothendieck spectral sequence, with E_2-page given by*

$$E_2^{i,j} = \mathrm{Ext}^i_{S(n,d)}(M, \mathrm{Ext}^j_{k\Sigma_d}(V^{\otimes d}, N)) \Rightarrow \mathrm{Ext}^{i+j}_{k\Sigma_d}(eM, N). \tag{2.2.1}$$

For $M \in M(n,d)$ and S a simple module in $M(n,d)$, let $[M : S]$ be the multiplicity of S as a composition factor of M. The following results from [7] provide information on composition factors of the higher right derived functors of $\mathcal{G}$.

Theorem (B). *Let $N \in \mathrm{mod}(k\Sigma_d)$, and $\mu \in \Lambda$. Then*

(i) $[R^j\mathcal{G}(N) : L(\mu)] = \dim_k \mathrm{Ext}^j_{k\Sigma_d}(Y^\mu, N)$ *for* $j \geq 0$.

(ii) *In particular, $e(R^j\mathcal{G}(N)) = 0$ for $j > 0$.*

2.3 A standard spectral sequence argument yields the following useful result.

Proposition . *Let $n \geq d$, $M \in M(n,d)$, and $N \in \mathrm{Mod}(k\Sigma_d)$ Suppose that $R^j\mathcal{G}(N) = 0$ for $1 \leq j \leq t$. Then*

$$\mathrm{Ext}^i_{S(n,d)}(M, \mathcal{G}(N)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, N)$$

for $0 \leq i \leq t$.

Proof. Consider the spectral sequence (2.2.1). By assumption, $\mathrm{Ext}^j_{k\Sigma_d}(V^{\otimes d}, N) = R^j\mathcal{G}(N) = 0$ for $1 \leq j \leq t$. Therefore, $E_2^{i,j} = 0$ if $j > 0$ and $1 \leq i+j \leq t$. The spectral sequence has differentials d_r with bidegree $(r, 1-r)$. Therefore,

$$E_2^{i,0} \cong \mathrm{Ext}^i_{S(n,d)}(M, \mathcal{G}(N)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, N)$$

for $0 \leq i \leq t$. □

3 Twisted Specht and Young modules

In this section we prove that the image under the functor $\mathcal{G}$ of a twisted Young (resp. Specht) module is a tilting (resp. Weyl) module as long as $p > 3$. These results for twisted Young modules can be viewed as dual to the results proved in [3, 5.2.4]. The result in [3] holds for $p > 2$ and can be used to prove its dual version (see [8, 6.2]). But we employ a different approach than the one given in [3].

3.1 We deal with Young modules first.

Lemma . *If $p > 3$ and $\lambda \in \Lambda$ then $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$ has a good filtration.*

Proof. Let $M = \Delta(\mu)$ and $N = Y^{\lambda'} \otimes \mathrm{sgn}$. From the five term exact sequence for the spectral sequence (2.2.1), we have

$$0 \to \mathrm{Ext}^1_{S(n,d)}(\Delta(\mu), \mathcal{G}(Y' \otimes \mathrm{sgn})) \hookrightarrow \mathrm{Ext}^1_{k\Sigma_d}((S^\mu)^*, Y^{\lambda'} \otimes \mathrm{sgn}).$$

However,

$$\mathrm{Ext}^1_{k\Sigma_d}((S^\mu)^*, Y^{\lambda'} \otimes \mathrm{sgn}) \cong \mathrm{Ext}^1_{k\Sigma_d}(Y^{\lambda'}, S^\mu \otimes \mathrm{sgn}) \cong \mathrm{Ext}^1_{k\Sigma_d}(Y^{\lambda'}, (S^{\mu'})^*).$$

As $Y^{\lambda'}$ is a summand of the permutation module $M^{\lambda'}$, the Frobenius reciprocity implies that $\mathrm{Ext}^1_{k\Sigma_d}(Y^{\lambda'}, (S^{\mu'})^*)$ is a summand of $\mathrm{Ext}^1_{k\Sigma_{\lambda'}}(k, (S^{\mu'})^*\downarrow_{\Sigma_{\lambda'}})$. Moreover, the restriction $(S^{\mu'})^*\downarrow_{\Sigma_{\lambda'}}$ has a Specht filtration by [13], so the last Ext-group is zero in view of [2, 2.4] or [8, 4.2]. Hence, $\mathrm{Ext}^1_{S(n,d)}(\Delta(\mu), \mathcal{G}(Y' \otimes \mathrm{sgn})) = 0$ for all $\mu \in \Lambda$, thus $\mathcal{G}(Y' \otimes \mathrm{sgn})$ has a good filtration. □

Theorem . *If $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$ has a good filtration then $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn}) = T(\lambda)$. In particular for $p > 3$ we have $\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn}) = T(\lambda)$ for all $\lambda \in \Lambda$.*

Proof. As $d \leq n$, every standard module has a p-restricted socle [12]. Hence $T(\lambda)$ also has a p-restricted socle. So by [7, 3.3] there exists an exact sequence of the form

$$0 \to T(\lambda) \to \mathcal{G}(Y' \otimes \mathrm{sgn}) \to X \to 0$$

where all composition factors of X are not p-restricted. Since $T(\lambda)$ and $\mathcal{G}(Y' \otimes \mathrm{sgn})$ have good filtrations, it follows that X must have a good filtration. Therefore, $X = 0$ because each $\nabla(\mu)$ has a p-restricted head by [12] again. □

3.2 We now prove that the image under $\mathcal{G}$ of a twisted Specht module is a Weyl module.

Theorem . *Let $p > 3$. Then $\Delta(\lambda) = \mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn})$ for all $\lambda \in \Lambda$.*

Proof. By [12], $\Delta(\lambda)$ has a p-restricted socle so in view of [7, 3.3] there exists a short exact sequence of the form

$$0 \to \Delta(\lambda) \to \mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) \to X \to 0.$$

Therefore, it will suffice to prove that $\dim_k \Delta(\lambda) = \dim_k \mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn})$ or equivalently that $\Delta(\lambda)$ and $\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn})$ have the same composition factors with multiplicities. According to Theorems 2.2B(i), 3.1 and [6, (3.8)], we have

$$\begin{aligned}
[\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) : L(\mu)] &= \dim_k \mathrm{Hom}_{k\Sigma_d}(Y^\mu, S^{\lambda'} \otimes \mathrm{sgn}) \\
&= \dim_k \mathrm{Hom}_{k\Sigma_d}((S^{\lambda'})^*, Y^\mu \otimes \mathrm{sgn}) \\
&= \dim_k \mathrm{Hom}_{S(n,d)}(\Delta(\lambda'), \mathcal{G}(Y^\mu \otimes \mathrm{sgn})) \\
&= \dim_k \mathrm{Hom}_{S(n,d)}(\Delta(\lambda'), T(\mu')) \\
&= [T(\mu') : \nabla(\lambda')] \\
&= [\Delta(\lambda) : L(\mu)].
\end{aligned}$$

□

3.3 For $p > 2$, $\mathrm{Ext}^1(Y^\lambda, Y^\mu) = 0$ for $\lambda, \mu \in \Lambda$ by [3, 4.6.1] or [8, 6.3]. The following theorem deals with extensions of Specht modules and extensions between Specht and Young modules. We note that part (a) was first observed in [3, 3.8.1].

Theorem . *Let $p > 3$, $\mu \in \Lambda_{\mathrm{res}}$, and $\lambda \in \Lambda$.*

(a) $\mathrm{Ext}^1_{k\Sigma_d}(S^\mu, Y^\lambda) = 0$;

(b) $\mathrm{Ext}^1_{k\Sigma_d}(S^\mu, S^\lambda) \cong \mathrm{Ext}^1_{S(n,d)}(\Delta(\mu'), \Delta(\lambda'))$;

(c) *If μ does not strictly dominate λ then* $\mathrm{Ext}^1_{k\Sigma_d}(S^\mu, S^\lambda) = 0$.

(d) *In particular,* $\mathrm{Ext}^1_{k\Sigma_d}(S^\mu, S^\mu) = 0$.

Proof. According to [7, 2.4A(ii)], one can let $M = \Delta(\mu')$ to get

$$\mathrm{Ext}^1_{k\Sigma_d}(S^\mu \otimes \mathrm{sgn}, N) \cong \mathrm{Ext}^1_{k\Sigma_d}((S^{\mu'})^*, N) \cong \mathrm{Ext}^1_{k\Sigma_d}(\Delta(\mu'), \mathcal{G}(N)). \tag{3.3.1}$$

For part (a) set $N = Y^\lambda \otimes \mathrm{sgn}$ and use Theorem 3.1. Part (b) follows by setting $N = S^\lambda \otimes \mathrm{sgn}$ and using Theorem 3.2. Note that if μ does not strictly dominate λ then λ' does not strictly dominate μ', and $\mathrm{Ext}^1_{S(n,d)}(\Delta(\mu'), \Delta(\lambda')) = 0$ by [14, II 2.14 (3)] proving part (c), of which (d) is a special case. □

4 Completely splittable modules

4.1 Definition. [15] A simple $k\Sigma_d$-module D is called *completely splittable* if the restriction $D{\downarrow}_{\Sigma_\mu}$ is completely reducible for any Young subgroup Σ_μ of Σ_d.

4.2 We now recall two basic results on CS modules proved in [15]. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0)$ define

$$h(\lambda) := s \quad \text{and} \quad \chi(\lambda) := \lambda_1 - \lambda_s + s.$$

The invariant $h(\lambda)$ is often referred to as the *height* of the partition. The first theorem provides necessary and sufficient conditions for a simple module to be CS.

Theorem (A). [15] *Let $\lambda \in \Lambda_{\mathrm{reg}}$. Then D^λ is completely splittable if and only if $\chi(\lambda) \leq p$.*

For a removable node A of a partition $\lambda \vdash n$ we denote by λ_A the partition of $n-1$ whose Young diagram is obtained from that of λ by removing A. The second result describes how CS representations decompose upon restriction to Σ_{d-1}.

Theorem (B). [15] *Let λ be a p-regular partition of d with $\chi(\lambda) \leq p$. Then $D^\lambda\downarrow_{\Sigma_{d-1}} \cong \bigoplus D^{\lambda_A}$ where the sum is over all removable nodes A of λ such that $\chi(\lambda_A) \leq p$.*

4.3 The following is an easy consequence of the branching rule given in Theorem 4.2B.

Proposition . *Let $m > d$, μ be any p-regular partition of d, and λ be a p-regular partition of m satisfying $h(\lambda) \leq s \quad \text{and} \quad \lambda_1 - \lambda_s + s \leq p$. Then D^μ appears as a composition factor of $D^\lambda\downarrow_{\Sigma_d}$ only if $h(\mu) \leq s \quad \text{and} \quad \mu_1 - \mu_s + s \leq p$.*

Proof. If $h(\lambda) = s$ then $\chi(\lambda) = \lambda_1 - \lambda_s + s \leq p$, and by Theorem 4.2A, D^λ is CS. If $h(\lambda) < s$ then $\lambda_s = 0$, and $\chi(\lambda) < \lambda_1 - \lambda_s + s \leq p$. So by Theorem 4.2A again, D^λ is CS in this case, too. Now apply Theorem 4.2B. □

4.4 A $k\Sigma_d$-module is called *non-trivial* if it is not isomorphic to the trivial module k. Our next result shows that non-trivial CS modules enjoy the remarkable property that they always give simple $S(n,d)$-modules upon application of the functor $\mathcal{G}$.

Theorem . *Let D_λ be a non-trivial completely splittable $k\Sigma_d$-module and D_μ be an arbitrary simple $k\Sigma_d$-module. Then*

(a) $\mathcal{G}(D_\lambda) = L(\lambda)$.
(b) *If $n \geq d$ then* $\mathrm{Ext}^1_{k\Sigma_d}(D_\mu, D_\lambda) \cong \mathrm{Ext}^1_{S(n,d)}(L(\mu), L(\lambda))$.

Proof. (a) According to Theorem 2.2B(i), we have $[\mathcal{G}(D_\lambda) : L(\nu)] = \dim_k \mathrm{Hom}_{k\Sigma_d}(Y^\nu, D_\lambda)$. If ν is p-restricted then Y^ν is a projective $k\Sigma_d$-module with simple head D_ν. So the only p-restricted composition factor of $\mathcal{G}(D_\lambda)$ is $L(\lambda)$. Now assume that ν is not p-restricted. Pick $\gamma \in \Lambda_{\mathrm{reg}}$ so that $D_\lambda = D^\gamma$. Then Y^ν is a direct summand of M^ν and

$$\mathrm{Hom}_{k\Sigma_d}(Y^\nu, D^\gamma) \subseteq \mathrm{Hom}_{k\Sigma_d}(M^\nu, D^\gamma) \cong \mathrm{Hom}_{k\Sigma_\nu}(k, D^\gamma\downarrow_{k\Sigma_\nu}).$$

We claim that the last space is zero. Indeed, $\Sigma_\nu \cong \Sigma_{\nu_1} \times \Sigma_{\nu_2} \times \dots$, with $\nu_1 \geq p$ because ν is not p-restricted. As D^γ is non-trivial, it follows that $s := h(\gamma) \geq 2$. Now by Proposition 4.3, $D^\gamma\downarrow_{k\Sigma_{\nu_1}}$ does not contain a trivial component. This completes the proof of part (a).

(b) By [7, 4.2(i)], we have $\mathrm{Ext}^1_{k\Sigma_d}(D_\mu, D_\lambda) \cong \mathrm{Ext}^1_{S(n,d)}(L(\mu), \mathcal{G}(D_\lambda))$. But by part (a), $\mathcal{G}(D_\lambda) = L(\lambda)$, which completes the proof. □

Martin and the first author conjectured that for any simple module D, $\mathrm{Ext}^1_{k\Sigma_d}(D, D) = 0$ as long as $p > 2$. The next result shows this holds for CS representations. It is a special case of [16, 2.10 and Remark (iv) on page 2], which show that the conjecture is true for $D = D^\lambda$ provided $h(\lambda) \leq p - 1$. However, we give a proof as our methods here are very different.

Corollary . *Let $p \geq 3$ and D be any CS module over $k\Sigma_d$. Then $\mathrm{Ext}^1_{k\Sigma_d}(D, D) = 0$.*

Proof. If $D_\lambda \not\cong k$ then $\mathrm{Ext}^1_{k\Sigma_d}(D_\lambda, D_\lambda) = \mathrm{Ext}^1_{S(n,d)}(L(\lambda), L(\lambda)) = 0$ by Theorem 4.4b and [14, II.2.14]. On the other hand, if $D_\lambda \cong k$ then $\mathrm{Ext}^1_{k\Sigma_d}(k, k) \cong \mathrm{H}^1(\Sigma_d, k) = 0$ as $p > 2$. □

5 Vanishing results

Theorem 4.4 motivates us to study images of CS modules under the higher derived functors of the functor $\mathcal{G}$. A vanishing result depending on the height will be proved in Section 5.4. But first we need three lemmas on symmetric group cohomology.

5.1 First we calculate cohomology groups of Σ_p with coefficients in simple modules.

Lemma . *The cohomology of Σ_p with coefficients in simple modules is given as follows.*

(i) $\mathrm{H}^j(\Sigma_p, D^\lambda) = 0$ *unless λ is of the form $(p - i, 1^i)$ for $0 \leq i \leq p - 2$.*

(ii) *For* $1 \leq i \leq p-2$,

$$\mathrm{H}^j(\Sigma_p, D^{(p-i,1^i)}) \cong \begin{cases} k & \textit{if } j \textit{ is of the form } 2m(p-1)+i \textit{ or} \\ & 2m(p-1)+(2p-i-3) \textit{ for some integer } m \geq 0; \\ 0 & \textit{otherwise.} \end{cases}$$

(iii) *For* $i = 0$,

$$\mathrm{H}^j(\Sigma_p, k) \cong \begin{cases} k & \textit{if } j \textit{ is of the form } 2m(p-1) \textit{ or } 2m(p-1)-1 \\ & \textit{for some integer } m \geq 0; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let P_i be the projective cover of $D^{(p-i,1^i)}$, $0 \leq i \leq p-2$. The module P_i admits a Specht filtration with factors $S^{(p-i,1^i)}, S^{(p-i-1,1^{i+1})}$ (starting from the top). Since P_i is self-dual it also has a filtration with factors $(S^{(p-i-1,1^{i+1})})^*, (S^{(p-i,1^i)})^*$. Therefore, a minimal projective resolution can be constructed as follows:

$$0 \leftarrow k \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{p-2} \leftarrow P_{p-2} \leftarrow P_{p-3} \leftarrow \cdots \leftarrow P_0 \leftarrow \ldots$$

Since the resolution is minimal and $D^{(p-i,1^i)}$ is a simple module, we have $\mathrm{H}^j(\Sigma_p, D^{(p-i,1^i)}) \cong \mathrm{Hom}_{k\Sigma_p}(Q_j, D^{(p-i,1^i)})$, where Q_j is the jth term of the resolution. The lemma follows. □

5.2 We now prove a vanishing result on cohomology of Σ_{p^m} with coefficients in CS modules, using Lemma 5.1 as an induction base. Note that all simple modules for $k\Sigma_p$ are CS.

Lemma . *Let* $m \geq 1$ *and* D^λ *be a CS module of* Σ_{p^m}.

(a) *If* $D^\lambda \not\cong k$, $h(\lambda) \leq s$ *and* $\lambda_1 - \lambda_s + s \leq p$, *then* $\mathrm{H}^j(\Sigma_{p^m}, D^\lambda) = 0$ *for* $0 \leq j \leq s-2$.
(b) *If* $D^\lambda \cong k$ *then* $\mathrm{H}^j(\Sigma_{p^m}, k) = 0$ *for* $1 \leq j \leq 2(p-2)$.

Proof. We apply induction on m. For $m = 1$ the result follows from Lemma 5.1. Let $m > 1$. The Sylow p-subgroup $\mathrm{Syl}_p(\Sigma_{p^m})$ embeds into the wreath product $A := \Sigma_{p^{m-1}} \wr \Sigma_p < \Sigma_{p^m}$. Hence $\mathrm{H}^n(\Sigma_{p^m}, D^\lambda)$ embeds into $\mathrm{H}^n(A, D^\lambda)$. Note that $A \cong B \rtimes \Sigma_p$ where B is the Young subgroup $\Sigma_{((p^{m-1})^p)} \cong \Sigma_{p^{m-1}} \times \cdots \times \Sigma_{p^{m-1}} < \Sigma_{p^m}$.

Now we apply the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} \cong \mathrm{H}^i(\Sigma_p, \mathrm{H}^j(B, D^\lambda)) \Rightarrow \mathrm{H}^{i+j}(A, D^\lambda). \tag{5.2.1}$$

By the Kunneth formula, we get

$$\mathrm{H}^j(B, D^\lambda) \cong \bigoplus \bigoplus_{\substack{(j_1,\ldots,j_p)\in(\mathbb{Z}+)^p \\ j_1+\cdots+j_p=n}} \mathrm{H}^{j_1}(\Sigma_{p^{m-1}}, D_1) \otimes \cdots \otimes \mathrm{H}^{j_p}(\Sigma_{p^{m-1}}, D_p)$$

where the first sum is over all composition factors $D_1 \otimes \cdots \otimes D_p$ of the restriction $D^\lambda{\downarrow_B}$. Note that we have used the fact that the restriction is completely reducible. If $D^\lambda \cong k$ and $1 \leq j \leq 2(p-2)$ then by inductive hypothesis the last expression is zero. So assume that D^λ is as given in part (a) and $0 \leq j \leq s-2$. By Proposition 4.3, none of the modules D_q in a composition factor $D_1 \otimes \cdots \otimes D_p$ of $D^\lambda{\downarrow_B}$ is trivial. On the other hand, $j_1 + \cdots + j_p = j \leq s-2 \leq p-3$ implies that at least one of the indices j_q in the sum above must be 0. So for this q we have $\mathrm{H}^0(\Sigma_{p^{n-1}}, D_q) = 0$. This shows that $\mathrm{H}^j(B, D^\lambda) = 0$ under our assumptions.

Finally, we prove part (a). As $\mathrm{H}^j(B, D^\lambda) = 0$ for $0 \leq j \leq s-2$, it follows that $E_2^{i,j} = 0$ for $0 \leq i+j \leq s-2$ in the spectral sequence (5.2.1). The differentials in this spectral sequence have bidegree $(r, 1-r)$, so $\mathrm{H}^n(A, D^\lambda) = 0$ for $0 \leq n \leq s-2$. Part (b) can be proved by using a similar line of reasoning and the fact that in this case $H^j(B, D^\lambda) = 0$ for $1 \leq j \leq 2(p-2)$ and $E_2^{i,0} = H^i(\Sigma_p, H^0(G, k)) = H^i(\Sigma_p, k) = 0$ for $1 \leq i \leq 2(p-2)$. □

5.3 Now we get a general vanishing result on cohomology of CS modules.

Lemma . *Let D^λ be a CS module of Σ_d.*

(a) *If $D^\lambda \not\cong k$, $h(\lambda) \le s$ and $\lambda_1 - \lambda_s + s \le p$ then $\mathrm{H}^j(\Sigma_d, D^\lambda) = 0$ for $0 \le j \le s-2$.*
(b) *If $D^\lambda \cong k$ then $\mathrm{H}^j(\Sigma_d, k) = 0$ for $1 \le j \le 2(p-2)$.*

Proof. If $d = p^{i_1} + p^{i_2} + \dots p^{i_k} + r$ for $i_1 \ge i_2 \ge \dots \ge i_q > 0$ and $0 \le r < p$, then there is a Sylow p-subgroup of Σ_d contained in the Young subgroup $G := \Sigma_{(p^{i_1},\dots,p^{i_q})}$. This implies that $\mathrm{H}^j(\Sigma_d, D^\lambda)$ embeds into $\mathrm{H}^j(G, D^\lambda)$. By the Kunneth formula, we get

$$\mathrm{H}^j(G, D^\lambda) \cong \bigoplus \bigoplus_{\substack{(j_1,\dots,j_q)\in(\mathbb{Z}+)^q \\ j_1+\dots+j_q=j}} \mathrm{H}^{j_1}(\Sigma_{p^{i_1}}, D^{\mu(1)}) \otimes \dots \otimes \mathrm{H}^{j_q}(\Sigma_{p^{i_q}}, D^{\mu(q)})$$

where the first sum is over all composition factors $D^{\mu(1)} \otimes \dots \otimes D^{\mu(q)}$ of the restriction $D^\lambda{\downarrow_G}$ (we have used the fact that the restriction is completely reducible). Furthermore, by Proposition 4.3, each $\mu(l)$ satisfies satisfies $h(\mu(l)) \le s$ and $\mu(l)_1 - \mu(l)_s + s \le p$. So, by Lemma 5.2, $\mathrm{H}^j(G, D^\lambda) = 0$ for $0 \le j \le s-2$ if $D^\lambda \not\cong k$, and $\mathrm{H}^j(G, k) = 0$ for $1 \le j \le 2(p-2)$. □

5.4 Finally, we obtain an information on vanishing of the higher derived functors of $\mathcal{G}$.

Theorem . *Let D^λ be a CS module of Σ_d.*

(a) *If $D^\lambda \not\cong k$, $h(\lambda) \le s$ and $\lambda_1 - \lambda_s + s \le p$ then $R^j\mathcal{G}(D^\lambda) = 0$ for $1 \le j \le s-2$.*
(b) *If $D^\lambda \cong k$ then $R^j\mathcal{G}(k) = 0$ for $1 \le j \le 2(p-2)$.*

Proof. (a) Using the definition of $\mathcal{G}$ and the decomposition of $V^{\otimes d}$ as a $k\Sigma_d$-module, we have

$$R^j\mathcal{G}(D^\lambda) \cong \mathrm{Ext}^j_{k\Sigma_d}(V^{\otimes n}, D^\lambda) \cong \bigoplus \mathrm{Ext}^j_{k\Sigma_d}(M^\mu, D^\lambda) \cong \bigoplus \mathrm{Ext}^j_{k\Sigma_\mu}(k, D^\lambda{\downarrow_{\Sigma_\mu}})$$

where the sums are over all compositions μ of d. For $1 \le j \le s-2$ and a composition $\mu = (\mu_1, \dots, \mu_a)$ of d we prove that $\mathrm{Ext}^j_{k\Sigma_\mu}(k, D^\lambda{\downarrow_{\Sigma_\mu}})$. Let $D^{\lambda(1)} \otimes \dots \otimes D^{\lambda(a)}$ be a composition factor of $D^\lambda{\downarrow_{\Sigma_\mu}}$. According to Proposition 4.3, every partition $\lambda(i)$ satisfies $h(\lambda(i)) \le s$ and $\lambda(i)_1 - \lambda(i)_s + s \le p$. Now $\mathrm{Ext}^j_{k\Sigma_\mu}(k, D^{\lambda(1)} \otimes \dots \otimes D^{\lambda(a)}) = 0$ by Lemma 5.3a and the Kunneth formula. Hence, $\mathrm{Ext}^j_{k\Sigma_\mu}(k, D^\lambda{\downarrow_{\Sigma_\mu}}) = 0$.

(b) As in (a), $R^j\mathcal{G}(k) \cong \oplus \mathrm{Ext}^j_{k\Sigma_\mu}(k,k)$ where the sum is over all compositions μ of d. But, $\mathrm{Ext}^j_{k\Sigma_\mu}(k,k) = 0$ for $1 \le j \le 2(p-2)$ by the Kunneth formula and Lemma 5.3b. □

Remark . The sign representation is CS and non-trivial if $p > 2$. It is equal to some D^ε with $h(\varepsilon) = p-1$ if $d \ge p-1$ ([11, 24.5(iii)]). As $k\Sigma_d$ is semisimple for $d < p$, the theorem above implies

$$R^j\mathcal{G}(\mathrm{sgn}) = 0 \quad \text{for } 1 \le j \le p-3. \tag{5.4.1}$$

6 Applications

6.1 Stability of Extensions I. The algebra $S(n,d)$ is quasi hereditary and thus has finite global dimension. On the other hand, $k\Sigma_d$ has infinite global dimension. So there is no hope for the extension groups of these algebras to agree in all degrees. Nevertheless, often one can show that the extension groups for $S(n,d)$ and $k\Sigma_d$ coincide for a certain *range* of degrees.

Theorem . *Let $D_\lambda = D^\nu$ be a non-trivial CS module and let $M \in M(n,d)$.*

(a) *For $0 \le i \le h(\nu) - 2$ we have $\mathrm{Ext}^i_{S(n,d)}(M, L(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, D_\lambda)$.*
(b) *In particular for any $\mu \in \Lambda$ and $0 \le i \le h(\nu) - 2$ we have:*

(i) $\mathrm{Ext}^i_{S(n,d)}(L(\mu), L(\lambda)) \cong \begin{cases} \mathrm{Ext}^i_{k\Sigma_d}(D_\mu, D_\lambda) & \textit{if } \mu \in \Lambda_{\mathrm{res}}; \\ 0 & \textit{otherwise.} \end{cases}$

(ii) $\mathrm{Ext}^i_{S(n,d)}(\nabla(\mu), L(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(S^\mu, D_\lambda)$.

(iii) $\mathrm{Ext}^i_{S(n,d)}(\Delta(\mu), L(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}((S^\mu)^*, D_\lambda)$.

(iv) $\mathrm{Ext}^i_{S(n,d)}(T(\mu), L(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(Y^{\mu'}, D^{\lambda'})$.

(v) $\mathrm{Ext}^i_{k\Sigma_d}(Y^\mu, D_\lambda) \cong \begin{cases} k & \textit{if } \lambda = \mu \textit{ and } i = 0; \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (a) By Theorem 4.4a, $\mathcal{G}(D_\lambda) = L(\lambda)$, and by Theorem 5.4, $R^j\mathcal{G}(D_\lambda) = 0$ for $1 \leq j \leq h(\nu) - 2$. The result now follows from Proposition 2.3.

(b) follows from (a). Indeed, for part (ii) (resp. (iii)) set $M = \nabla(\mu)$ (resp. $M = \Delta(\mu)$) and using the results stated in Section 2.1. For (i) set $M = L(\mu)$ and use the fact that $eL(\lambda) = 0$ for $\lambda \notin \Lambda_{\mathrm{res}}$. To prove part (iv), set $M = T(\mu)$ and use (2.1.1). For part (v), set $M = P(\mu)$ where $P(\mu)$ is the projective cover of $L(\mu)$. Observe that $\mathrm{Ext}^i_{S(n,d)}(P(\mu), L(\lambda))$ is zero if $i > 0$ or $\lambda \neq \mu$ and isomorphic to k if $i = 0$ and $\lambda = \mu$. □

Remark. (a) Note that even in the case $i = 0$ the theorem above is saying something new. Part (v) implies that a non-trivial CS module D_λ appears in the head of a Young module Y^μ if *and only if* $\lambda = \mu$. If μ is not p-restricted, the head of Y^μ need not be simple.

(b) Parts (iv) and (v) of the theorem can be used to prove some vanishing results for $\mathrm{Ext}^i_{S(n,d)}(T(\mu), L(\lambda))$. We leave the formulation of the corresponding results to the reader.

Corollary . *Let $M \in M(d,d)$ and $\delta = L(1^d)$ be the determinant representation for $S(d,d)$.*

(a) *For $0 \leq i \leq p-3$, $\mathrm{H}^i(\Sigma_d, eM \otimes \mathrm{sgn}) \cong \mathrm{Ext}^i_{S(d,d)}(\delta, M)$.*

(b) *In particular for any $\mu \in \Lambda$ and $0 \leq i \leq p-3$ we have:*

(i) $\mathrm{H}^i(\Sigma_d, D^{\mu'}) \cong \mathrm{Ext}^i_{S(d,d)}(\delta, L(\mu))$ *for* $\mu \in \Lambda_{\mathrm{res}}$.

(ii) $\mathrm{H}^i(\Sigma_d, S^{\mu'}) \cong \mathrm{Ext}^i_{S(d,d)}(\delta, V(\mu))$.

(iii) $\mathrm{H}^i(\Sigma_d, (S^{\mu'})^*) \cong \begin{cases} k & \textit{if } \mu = (1^d) \textit{ and } i = 0; \\ 0 & \textit{otherwise.} \end{cases}$

(iv) $\mathrm{H}^i(\Sigma_d, Y^\mu \otimes \mathrm{sgn}) \cong \begin{cases} k & \textit{if } \mu = (1^d) \textit{ and } i = 0; \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (a) Let $D_\lambda = \mathrm{sgn}$. Then by Theorem 6.1a and (5.4.1) we have for $0 \leq i \leq p-3$,

$$\begin{aligned} \mathrm{H}^i(\Sigma_d, eM \otimes \mathrm{sgn}) &\cong \mathrm{Ext}^i_{k\Sigma_d}((eM)^*, \mathrm{sgn}) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM^\tau, \mathrm{sgn}) \\ &\cong \mathrm{Ext}^i_{S(n,d)}(M^\tau, \delta) \cong \mathrm{Ext}^i_{S(n,d)}(\delta, M). \end{aligned}$$

(b) By (2.1.1), $D^{\mu'} = D_\mu \otimes \mathrm{sgn}$, and $(S^{\mu'})^* = S^\mu \otimes \mathrm{sgn}$ so parts (i)-(iii) follow. Part (iv) follows from Theorem 6.1b(v). □

6.2 BKM conjecture. We now note that a conjecture made in [2, Conj. 6.2] (stated in the corollary below) is a rather special case of Corollary 6.1b(iii).

Corollary . *For a fixed $i > 0$ there exists a constant C depending only on i such that for $p > C$ we have $\mathrm{H}^i(\Sigma_d, (S^\lambda)^*) = 0$ for all d and $\lambda \in \Lambda$. In fact, we can take $C = i + 2$.*

Proof. Let $i > 0$. Set $C = i+2$. If $p > i+2$, or in other words $i < p-2$ then $\mathrm{H}^i(\Sigma_d, (S^\lambda)^*) = 0$ by Corollary 6.1b(iii) for all d and λ. □

6.3 Stability of Extensions II. We can improve the range of the stability of extensions as long as one of the modules involved is $\Delta(d) \cong S^d(V)^\tau$, the contravariant dual of the dth symmetric power of the natural module V.

Theorem . *Let $M \in M(n,d)$ with $n \geq d$.*

(a) *For $0 \leq i \leq 2(p-2)$ we have* $\mathrm{Ext}^i_{S(n,d)}(M, \Delta(d)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, k)$.

(b) *In particular for any $\mu \in \Lambda$ and $0 \leq i \leq 2(p-2)$ we have:*

(i) $\mathrm{Ext}^i_{S(n,d)}(L(\mu), \Delta(d)) \cong \begin{cases} \mathrm{Ext}^i_{k\Sigma_d}(D_\mu, k) & \text{if } \mu \in \Lambda_{\mathrm{res}}; \\ 0 & \text{otherwise.} \end{cases}$

(ii) $\mathrm{Ext}^i_{S(n,d)}(\nabla(\mu), \Delta(d)) \cong \mathrm{Ext}^i_{k\Sigma_d}(S^\mu, k)$.

(iii) $\mathrm{Ext}^i_{S(n,d)}(\Delta(\mu), \Delta(d)) \cong \mathrm{Ext}^i_{k\Sigma_d}((S^\mu)^*, k)$.

(iv) $\mathrm{Ext}^i_{S(n,d)}(T(\mu), \Delta(d)) \cong \mathrm{Ext}^i_{k\Sigma_d}(Y^{\mu'}, \mathrm{sgn})$.

(v) $\mathrm{Ext}^i_{k\Sigma_d}(Y^\mu, k) \cong \begin{cases} k & \text{if } i = 0 \text{ and } [\Delta(d) : L(\mu)] \neq 0; \\ 0 & \text{otherwise.} \end{cases}$

Proof. (a) We have $\mathcal{G}(k) = \Delta(d)$ by [7, 5.5] and $R^j\mathcal{G}(k) = 0$ for $1 \leq j \leq 2(p-2)$ by Theorem 5.4. Now use Proposition 2.3.

(b) Parts (i)-(v) follows from (a) by setting $M = \nabla(\mu)$ (resp. $\Delta(\mu)$, $T(\mu)$, $P(\mu)$) as in the proof of Theorem 6.1b. For (v) we also use the fact that $\Delta(d)$ is multiplicity free. □

Corollary . *Let $M \in M(n,d)$ with $n \geq d$.*

(a) *For $0 \leq i \leq 2(p-2)$ we have* $\mathrm{H}^i(\Sigma_d, eM) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), M)$.

(b) *In particular for any $\mu \in \Lambda$ and $0 \leq i \leq 2(p-2)$ we have:*

(i) $\mathrm{H}^i(\Sigma_d, D_\mu) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), L(\mu))$ *for* $\mu \in \Lambda_{\mathrm{res}}$.

(ii) $\mathrm{H}^i(\Sigma_d, (S^\mu)^*) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), \Delta(\mu))$.

(iii) $\mathrm{H}^i(\Sigma_d, S^\mu) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), \nabla(\mu))$.

(iv) $\mathrm{H}^i(\Sigma_d, Y^{\mu'} \otimes \mathrm{sgn}) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), T(\mu))$.

(v) $\mathrm{H}^i(\Sigma_d, Y^\mu) \cong \begin{cases} k & \text{if } i = 0 \text{ and } [\nabla(d) : L(\mu)] \neq 0; \\ 0 & \text{otherwise.} \end{cases}$

Proof. (a) By Theorem 6.3a, one has for $0 \leq i \leq 2(p-2)$

$$\begin{aligned} \mathrm{H}^i(\Sigma_d, eM) &\cong \mathrm{Ext}^i_{k\Sigma_d}(k, eM) \cong \mathrm{Ext}^i_{k\Sigma_d}((eM)^*, k) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM^\tau, k) \\ &\cong \mathrm{Ext}^i_{S(n,d)}(M^\tau, \Delta(d)) \cong \mathrm{Ext}^i_{S(n,d)}(\nabla(d), M). \end{aligned}$$

(b) For parts (i)-(iv) make the appropriate substitutions for M, (v) follows from Theorem 6.3b(v) above by dualizing. □

6.4 Stability of Extensions III. A conjecture of Burichenko, Kleshchev and Martin verified in Section 6.2 will now be used to obtain stability results involving Specht modules.

Theorem . *Let $p > 3$, $0 \leq i \leq p-3$, $M \in M(n,d)$, and $\lambda, \mu \in \Lambda$. Then*

(a) $\mathrm{Ext}^i_{S(n,d)}(M, \Delta(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, S^{\lambda'} \otimes \mathrm{sgn})$.

(b) *In particular,*

(i) $\mathrm{Ext}^i_{S(n,d)}(L(\mu), \Delta(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(D^{\mu'}, S^{\lambda'})$.

(ii) $\mathrm{Ext}^i_{S(n,d)}(\Delta(\mu), \Delta(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(S^{\mu'}, S^{\lambda'})$.

(iii) $\mathrm{Ext}^i_{S(n,d)}(\nabla(\mu), \Delta(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(S^\mu, S^{\lambda'} \otimes \mathrm{sgn})$.

(iv) $\mathrm{Ext}^i_{k\Sigma_d}(Y^\mu, S^\lambda) = 0$.

(v) $\mathrm{Ext}^i_{k\Sigma_d}(S^\lambda, Y^\mu) = 0$.

Proof. (a) By Theorem 3.2, $\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) = \Delta(\lambda)$. Moreover, by Frobenius reciprocity,

$$R^j\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) = \mathrm{Ext}^j_{k\Sigma_d}(V^{\otimes d}, S^{\lambda'} \otimes \mathrm{sgn}) = \oplus \mathrm{Ext}^j_{k\Sigma_\mu}(k, (S^\lambda)^* \downarrow_{\Sigma_\mu})$$

where the last sum is over all compositions μ of d. By [13], $(S^\lambda)^* \downarrow_{\Sigma_\mu}$ admits a filtration with sections of the form $(S^{\lambda(1)})^* \otimes (S^{\lambda(2)})^* \otimes \cdots \otimes (S^{\lambda(a)})^*$. By the Kunneth formula and Corollary 6.1b(iii), it follows that $\mathrm{Ext}^j_{k\Sigma_\mu}(k, (S^{\lambda(1)})^* \otimes (S^{\lambda(2)})^* \otimes \cdots \otimes (S^{\lambda(a)})^*) = 0$ for $1 \le j \le p-3$. Consequently, by using induction on the length of the filtration on $(S^\lambda)^*$, we have $R^j\mathcal{G}(S^{\lambda'} \otimes \mathrm{sgn}) = 0$ for $1 \le j \le p-3$. Part (a) now holds by applying Theorem 2.3.

(b) For (i)-(v) substitute $M = L(\mu)$, $\Delta(\mu)$, $\nabla(\mu)$, $T(\mu)$ and $P(\mu)$, respectively. □

6.5 Stability of Extensions IV. We now obtain stability results on extensions involving Young modules.

Theorem . *Let $p > 3$, $0 \le i \le p-3$, $M \in M(n,d)$, and $\lambda, \mu \in \Lambda$.*

(a) $\mathrm{Ext}^i_{S(n,d)}(M, T(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(eM, Y^{\lambda'} \otimes \mathrm{sgn})$.

(b) *In particular,*

(i) $\mathrm{Ext}^i_{S(n,d)}(L(\mu), T(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(D^{\mu'}, Y^{\lambda'})$.

(ii) $\mathrm{Ext}^i_{S(n,d)}(\nabla(\mu), T(\lambda)) \cong \mathrm{Ext}^i_{k\Sigma_d}(S^\mu \otimes \mathrm{sgn}, Y^{\lambda'}) = 0$.

(iii) $\mathrm{Ext}^i_{k\Sigma_d}(Y^\mu, Y^\lambda) = 0$.

(iv) $\mathrm{Ext}^i_{k\Sigma_d}(Y^\mu \otimes \mathrm{sgn}, Y^\lambda) = 0$.

Proof. (a) By Theorem 3.1, $T(\lambda) = \mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn})$. Furthermore, if μ is a composition of d then the restriction of $Y^{\lambda'} \otimes \mathrm{sgn}$ to Σ_μ admits a filtration with sections of the form $(Y^{\lambda(1)} \otimes \mathrm{sgn}) \otimes \cdots \otimes (Y^{\lambda(a)} \otimes \mathrm{sgn})$. Now use the same argument as in Theorem 6.4 along with Corollary 6.1b(iv) to show that $R^j\mathcal{G}(Y^{\lambda'} \otimes \mathrm{sgn}) = 0$ for $1 \le j \le p-3$. Finally apply Theorem 2.3.

(b) Substitute $M = L(\mu)$, $\Delta(\mu)$, $T(\mu)$ and $P(\mu)$, respectively. □

We remark that the results given in Theorem 6.4b(iv)-(v) and Theorem 6.5b(iii) can be viewed as natural generalizations involving higher extension groups of the Ext^1-results given in the work of Cline, Parshall and Scott [3, §3.8, §4.6].

6.6 Alternating group cohomology. Let A_d be the alternating group on d letters. Nakaoka computed the structure of the cohomology of the symmetric group $\mathrm{H}^\bullet(\Sigma_d, k)$ [18]. For the prime $p = 2$, there is a method to compute $\mathrm{H}^\bullet(A_d, k)$ by using the calculation of $\mathrm{H}^\bullet(\Sigma_d, k)$ [1]. The first proposition shows that one can compute the cohomology of the alternating group by knowing the cohomology of the symmetric group with coefficients in the trivial and sign representations.

Proposition . *Let $p \ge 3$. Then $\mathrm{H}^\bullet(A_d, k) \cong \mathrm{H}^\bullet(\Sigma_d, k) \oplus \mathrm{H}^\bullet(\Sigma_d, \mathrm{sgn})$ as k-vector spaces.*

Proof. Note that the module induced from the trivial kA_d-module to $k\Sigma_d$ is isomorphic to $k \oplus \mathrm{sgn}$. The result now follows from the Eckmann-Shapiro lemma. □

Corollary . *Let $p \ge 3$. Then $\mathrm{H}^i(A_d, k) = 0$ for $1 \le i \le p-3$.*

Proof. This follows from the proposition above, Lemma 5.3 and Remark 5.4. □

Remark . Consider the spectral sequence (2.2.1) with $M = S^d(V)$ and $N = \mathrm{sgn}$:

$$E_2^{i,j} = \mathrm{Ext}^i_{S(n,d)}(S^d(V), R^j\mathcal{G}(\mathrm{sgn})) \Rightarrow \mathrm{H}^{i+j}(\Sigma_d, \mathrm{sgn}).$$

This indicates that there may be some hope to compute the cohomology of the alternating group by determining $R^\bullet\mathcal{G}(\mathrm{sgn}) = \mathrm{Ext}^\bullet_{k\Sigma_d}(V^{\otimes d}, \mathrm{sgn})$ as a $\mathsf{GL}_n(k)$-module.

References

[1] A. Adem, J.R. Milgram, *Cohomology of finite groups*, Springer-Verlag, Berlin, 1994.

[2] V.P. Burichenko, A.S. Kleshchev, S. Martin, On cohomology of dual Specht modules, *J. Pure Appl. Alg.*, 112, no. 2, (1996), 440-480.

[3] E.T. Cline, B.J. Parshall, L.L. Scott, Stratifying Endomorphism Algebras, *Memoirs of AMS*, 124 , no. 591, (1996).

[4] E.T. Cline, B.J. Parshall, L.L. Scott, Generic and q rational representation theory, *Publ. RIMS* 35 (1999), 31–90.

[5] S. Donkin, On Schur algebras and related algebras I, *J. Algebra* 104 (1986), 310–328.

[6] S. Donkin, On tilting modules for algebraic groups, *Math. Z.* 212 (1993), 39–60.

[7] S. R. Doty, K. Erdmann, D. K. Nakano, Extensions of modules over Schur algebras, symmetric groups and Hecke algebras, *preprint (1999).*

[8] S. R. Doty, D. K. Nakano, Relating the cohomology of general linear groups and symmetric groups, *to appear in Proc. of a Conference on Modular Representation Theory, Charlottesville, VA* (1998).

[9] K. Erdmann, Decomposition numbers for symmetric groups and composition factors of Weyl modules, *J. Algebra*, 180, (1996), 316-320.

[10] J. A. Green, *Polynomial Representations of GL_n*, (Lecture Notes in Math. **830**), Springer-Verlag, New York 1980.

[11] G.D. James, *The representation theory of the symmetric group,* Lecture Notes in Mathematics, 682, Springer 1978

[12] G.D. James, The decomposition of tensors over fields of prime characteristic, *Math. Z.* 172 (1980), 161-178.

[13] G. D. James and M. H. Peel, Specht series for skew representations of symmetric groups, *J. Algebra*, **56** (1979), 343–364.

[14] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando 1987.

[15] A. S. Kleshchev, Completely splittable representations of symmetric groups, *J. Algebra* 181 (1996), 584–592.

[16] A. S. Kleshchev and J. Sheth, On extensions of simple modules over symmetric and algebraic groups, *J. Algebra*, to appear.

[17] O. Mathieu, On the dimension of some modular irreducible representations of the symmetric group, *Lett. Math. Phys.* 38 (1996), 23-32.

[18] M. Nakaoka, Homology of the infinite symmetric group, *Ann. Math.* 73 (1961), 16-42.

[19] A. Ryba, Fibonacci representations of the symmetric groups, *J. Algebra* 170 (1994), 678–686.

[20] H. Wenzl, Hecke algebras of type A_n and subfactors, *Invent. Math.* 92 (1988), 349-383.

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