BRANCHING RULES FOR SYMMETRIC GROUPS AND APPLICATIONS

A.S. KLESHCHEV
Department of Mathematics
University of Oregon
Eugene, OR 97403
USA
E-mail: klesh@math.uoregon.edu

1. Introduction

Let F be an arbitrary field of characteristic $p \geq 0$, and Σ_n be the symmetric group on n letters. We shall discuss finite dimensional representations of Σ_n over F or, equivalently, finite dimensional $F\Sigma_n$-modules. Basic facts important for our topic can be found in [18, 21, 20]. We recall here some of them.

An explicit (though complicated) construction associates to every partition λ of n (written $\lambda \vdash n$) a $\mathbb{Z}\Sigma_n$-module $S^\lambda_\mathbb{Z}$. This module is free as a $\mathbb{Z}$-module. So the dimension of the $F\Sigma_n$-module $S^\lambda := S^\lambda_\mathbb{Z} \otimes_\mathbb{Z} F$ is independent of F. This module is called the *Specht module* (over F).

The properties of the Specht modules depend on the characteristic p of the ground field F. If $p = 0$, all modules S^λ are irreducible. Moreover, $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of the irreducible $F\Sigma_n$-modules (up to isomorphism). If $p > 0$, the modules S^λ are not irreducible in general. The following result shows that if S^λ is not 1-dimensional then it is *reducible* in *some* characteristic.

Proposition 1.1 [33] *If $\lambda \neq (n), (1^n)$ then there exists a prime p such that $S^\lambda_\mathbb{Z} \otimes_\mathbb{Z} F$ is reducible for any field F of characteristic p.*

However, the modules S^λ can be used to get an information on the irreducible $F\Sigma_n$-modules. Recall that a partition λ is called *p-regular* if no part in λ is repeated p or more times. It was proved by James that if λ is p-regular then S^λ has a simple head, say D^λ, and that

$$\{D^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } p\text{-regular}\}$$

is a complete set of the irreducible $F\Sigma_n$-modules up to isomorphism.

The fact that D^λ is the simple head of S^λ, being very useful, does not provide answers to many important questions about the irreducible modules. For example, the dimension of D^λ is not known in general.

Dimension problem. Find $\dim D^\lambda$.

Remark. Carter's criterion proved by James and Murphy [17, 22] describes when $D^\lambda = S^\lambda$. In this case we of course know the dimension of D^λ from the hook formula. For λ satisfying the Carter's criterion we also know the dimension of $D^\lambda \otimes \mathrm{sgn}$. The Young diagram corresponding to $D^\lambda \otimes \mathrm{sgn}$ is given by the Mullineux bijection, see Section 5. Mathieu [35] found another special class of the irreducibles whose dimensions can be described (see also [31] and Section 6 of this paper). Some asymptotic results [19] and lower bounds (see [29, 32] and (9) in Section 6) are also available. For dimensions of the irreducible modules corresponding to the two-part partitions see [8].

Define the *decomposition numbers* $d_{\lambda\mu}$ for Σ_n by setting

$$d_{\lambda\mu} = d^{(n)}_{\lambda\mu} = \left[S^\lambda : D^\mu\right]_{\Sigma_n},$$

the multiplicity of D^μ as a composition factor of S^λ. It is well known [18, 12.2] that $d_{\lambda\lambda} = 1$ (for any p-regular λ) and $d_{\lambda\mu} = 0$ unless μ dominates λ. Hence

$$\dim D^\lambda = \dim S^\lambda - \sum_{\mu \rhd \lambda} d_{\lambda\mu} \dim D^\mu.$$

So the knowledge of the decomposition numbers $d_{\lambda\mu}$ would provide us with an inductive algorithm for computing the dimensions of the irreducible modules D^λ, the inductive base being $\lambda = (n)$ where $\dim D^{(n)} = 1$.

Thus we have an even more general

Decomposition numbers problem for symmetric groups. Find the decomposition numbers $d^{(n)}_{\lambda\mu}$ for all (admissible) λ, μ and n.

This is the central problem in the modular representation theory of Σ_n. This problem is also equivalent to a similar problem for simple algebraic groups of type A. To be more precise, assume for the moment that F is infinite and denote by $L(\lambda)$ the irreducible polynomial representation of the algebraic group $GL_n(F)$ with highest weight λ, see e.g. [15]. Then λ can be identified with a partition of a non-negative integer r with at most n parts. The central problem of the representation theory of $GL_n(F)$ is to find the formal characters of the irreducible modules $L(\lambda)$. This problem is equivalent to the problem of determining the "decomposition numbers"

$$f_{\lambda\mu} = f^{(n)}_{\lambda\mu} = [V(\lambda) : L(\mu)]_{GL_n(F)}$$

where $V(\lambda)$ denotes the Weyl module over $GL_n(F)$ with highest weight λ. The relation between the decomposition numbers for the general linear and the symmetric groups is as follows.

Denote by λ' the partition whose Young diagram is the transpose of the Young diagram of λ. We say that λ' is the partition *conjugate* to λ.

Theorem 1.2 [15, 6.3,6.4,6.5]. *Let λ and μ be partitions of n, μ being p-regular. Then for any $N \geq n$ we have $d^{(n)}_{\lambda\mu} = f^{(N)}_{\lambda'\mu'}$.*

This result shows that the knowledge of the decomposition numbers for $GL_n(F)$ would give us the knowledge of the decomposition numbers for Σ_n. The fact that the converse is in a sense true was recently proved by Erdmann. If ν is a dominant weight for $GL_n(F)$ (i.e. a partition of $r \in \mathbb{Z}_{\geq 0}$ with at most n parts) denote by $t(\nu)$ the dominant weight

$$t(\nu) = p\nu + (p-1)(n-1, n-2, \dots, 1, 0).$$

Note that $t(\nu)$ is always p-regular.

Theorem 1.3 [9] $f_{\lambda\mu} = d_{t(\lambda'),t(\mu')}$.

This result shows that the decomposition numbers for $GL_n(F)$ are equal to certain decomposition numbers for Σ_N when N is allowed to vary. Another interesting corollary of Theorem 1.3 is as follows: to find *all* decomposition numbers for the symmetric groups it suffices to find only decomposition numbers $d_{\lambda\mu}$ with λ and μ in special blocks, namely the blocks corresponding to the cores of the form $(p-1)(k, k-1, \dots, 1, 0)$ (see [21, 2.7]).

Thus the problem of determining the decomposition numbers for the symmetric groups and the problem of determining the formal characters of the irreducible modules over general linear groups are equivalent.

In this paper we are going to concentrate on another problem which is equivalent to these two problems:

Branching problem. Determine the branching multiplicities

$$b_{\lambda\mu} := \left[D^\lambda \downarrow_{\Sigma_{n-1}} : D^\mu\right].$$

Here λ and μ are arbitrary p-regular partitions of n and $n-1$, respectively.

We shall show in Section 3 that, indeed, the numbers $d_{\lambda\mu}$ determine the numbers $b_{\lambda\mu}$ and vice versa.

Let P^ν denote the projective cover of D^ν. If μ is a p-regular partition of $n-1$ then the induced module $P^\mu\uparrow^{\Sigma_n}$ is projective again. We claim that

$$P^\mu\uparrow^{\Sigma_n}\cong\bigoplus_\lambda b_{\lambda\mu}P^\lambda$$

where the sum is over all p-regular partitions λ of n. Indeed, this easily follows from the following equalities

$$b_{\lambda\mu}=\dim\mathrm{Hom}_{\Sigma_{n-1}}(P^\mu, D^\lambda\downarrow_{\Sigma_{n-1}})=\dim\mathrm{Hom}_{\Sigma_n}(P^\mu\uparrow^{\Sigma_n}, D^\lambda).$$

Thus the branching coefficients $b_{\lambda\mu}$ can also be interpreted in terms of inducing the projective modules.

Finally note that the problem on the composition factors of the induced module $D^\mu\uparrow^{\Sigma_n}$ is connected with but *not equivalent* to the Branching problem, so it should be considered separately.

2. Some definitions and notation

Fix an arbitrary partition $\lambda=(\lambda_1\geq\lambda_2\geq\ldots)\vdash n$. We do not distinguish between λ and its Young diagram

$$\lambda=\{(i,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\mid j\leq\lambda_i\}.$$

Elements $(i,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ are called *nodes*. The *(p-)residue*, $\mathrm{res}A$, of a node $A=(i,j)$ is defined to be the (p-)residue class $(j-i) \pmod p$.

A node $(i,\lambda_i)\in\lambda$ is called a *removable node* (of λ) if $\lambda_i>\lambda_{i+1}$. A node (i,λ_i+1) is called an *addable node* (for λ) if $i=1$ or $i>1$ and $\lambda_i<\lambda_{i-1}$. The removable (resp., addable) nodes are the nodes which can be removed from (resp., added to) the Young diagram λ to produce a Young diagram with $n-1$ (resp., $n+1$) nodes.

If $A=(i,\lambda_i)$ is a removable node then

$$\lambda_A:=\lambda\setminus\{A\}=(\lambda_1,\ldots,\lambda_{i-1},\lambda_i-1,\lambda_{i+1},\ldots)$$

is a partition of $n-1$ obtained from λ by removing A. Similarly, if $B=(i,\lambda_i+1)$ is an addable node then

$$\lambda^B=\lambda\cup\{B\}=(\lambda_1,\ldots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},\ldots)$$

is a partition of $n+1$ obtained from λ by adding B.

A removable node A of λ is called *normal* if for every addable node B above A with $\mathrm{res}B=\mathrm{res}A$ there exists a removable node $C(B)$ strictly between A and B with $\mathrm{res}C(B) = \mathrm{res}A$, and $B \neq B'$ implies $C(B) \neq C(B')$. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. An addable node B is called *good* if it is good as a removable node of λ^B.

We denote by sgn the one dimensional sign representation of Σ_n.

The *residue content* of the Young diagram λ is defined to be the p-tuple

$$\mathrm{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1})$$

where

$$c_\alpha = \sharp\{\text{nodes in } \lambda \text{ of residue } \alpha\}, \ \alpha = 0, 1, \dots, p-1.$$

A well known result called the Nakayama Conjecture [21, 2.7.41, 6.1.21] claims that D^λ and D^μ are in one block if and only if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$. (One should be a little careful with the reference [21, 6.1.21] here. There p-blocks of the *ordinary* irreducible representations are considered. So one also have to use the fact that D^λ is a composition factor of S^λ.)

3. Some history and generalities

If we restrict S^λ, rather than D^λ, to Σ_{n-1} then the situation is usually treated as well understood, because of the following well known result which we call the Classical branching theorem (see e.g. [18, 9.3]).

Theorem 3.1 (Classical branching theorem). *Let*

$$A_1, A_2, \dots, A_k$$

be the removable nodes of λ counted from top to bottom. Then there exists a filtration

$$(0) = S_0 \subset S_1 \subset \dots \subset S_k = S^\lambda \downarrow_{\Sigma_{n-1}}$$

of the Σ_{n-1}-module $S^\lambda \downarrow_{\Sigma_{n-1}}$ such that $S_i/S_{i-1} \cong S^{\lambda_{A_i}}$, $i = 1, 2, \dots, k$.

In the characteristic 0 case this result immediately yields the classical branching rule. If $p > 0$ we note that even though the factors of the filtration are described, the submodule structure of $S^\lambda \downarrow_{\Sigma_{n-1}}$ is very complicated. For example the socle of this module is not known.

The "dual" result is as follows.

Theorem 3.2 [18, 17.14]. *Let $B_1, B_2, \dots, B_l$ be the addable nodes for λ counted from bottom to top. Then there exists a filtration*

$$(0) = T_0 \subset T_1 \subset \dots \subset T_l = S^\lambda \uparrow^{\Sigma_{n+1}}$$

such that $T_i/T_{i-1} = S^{\lambda^{B_i}}$, $i = 1, 2, \dots, l$.

The first non-trivial result on the restriction $D^\lambda\downarrow_{\Sigma_{n-1}}$ was obtained by James in 1976.

Theorem 3.3 [16]. *Let*

$$A_1, A_2, \dots, A_k$$

be the removable nodes of a p-regular partition λ counted from top to bottom. Set $m \geq 1$ to be the minimal index such that λ_{A_m} is p-regular. Then $\left[D^\lambda\downarrow_{\Sigma_{n-1}}: D^\mu\right] \neq 0$ implies $\mu \trianglerighteq \lambda_{A_m}$, and $\left[D^\lambda\downarrow_{\Sigma_{n-1}}: D^{\lambda_{A_m}}\right] = m$.

This result shows in particular that $D^\lambda\downarrow_{\Sigma_{n-1}}$ is not multiplicity free in general. Moreover the multiplicities can take *any* non-negative integral values.

Now we are in a position to explain why the Branching problem is equivalent to the Decomposition numbers problem.

Assume first that the decomposition numbers $d^{(n)}_{\lambda\mu}$ are known for all n. We find the numbers $b_{\lambda\mu}$ by induction on λ. If $\lambda = (n)$ then $D^{(n)}$ is the trivial module, and everything is clear. Let $\lambda \neq (n)$, and suppose the numbers $b_{\nu\kappa}$ are known for all $\nu \rhd \lambda$. In the Grothendieck group we can write

$$S^\lambda\downarrow_{\Sigma_{n-1}} = \sum_{i=1}^{k} S^{\lambda_{A_i}} = \sum_{i=1}^{k}\sum_{\mu} d_{\lambda_{A_i},\mu} D^\mu, \tag{1}$$

i.e. the composition factors of $S^\lambda\downarrow_{\Sigma_{n-1}}$ are known. On the other hand, $S^\lambda = D^\lambda + \sum_{\nu\rhd\lambda} d_{\lambda\nu} D^\nu$. So

$$S^\lambda\downarrow_{\Sigma_{n-1}} = D^\lambda\downarrow_{\Sigma_{n-1}} + \sum_{\nu\rhd\lambda} d_{\lambda\nu} D^\nu\downarrow_{\Sigma_{n-1}},$$

whence

$$D^\lambda\downarrow_{\Sigma_{n-1}} = S^\lambda\downarrow_{\Sigma_{n-1}} - \sum_{\nu\rhd\lambda} d_{\lambda\nu} D^\nu\downarrow_{\Sigma_{n-1}}.$$

But the second term of the RHS is also known (by the inductive hypothesis).

Conversely, assume that the numbers $b_{\lambda\mu}$ are known for all n. We demonstrate how to find the numbers $d_{\lambda\mu}$ inductively. Assume $d_{\lambda\mu}$ are known for Σ_{n-1}. To find the decomposition numbers for Σ_n, note that the composition factors of the restriction $S^\lambda\downarrow_{\Sigma_{n-1}}$ are known, see (1), since $d_{\lambda_{A_i},\mu}$ are decomposition numbers for Σ_{n-1}. Thus the question on the composition factors of S^λ (equivalently the decomposition numbers $d_{\lambda\mu}$ with a fixed λ) can be reduced to the following

Questions. Let M be an $F\Sigma_n$-module. Do the composition factors of the restriction $M\downarrow_{\Sigma_{n-1}}$ (together with their multiplicities) determine uniquely the composition factors of M (together with their multiplicities)? If yes, is there an algorithm for determining the composition factors of M, providing the branching coefficients $b_{\lambda\mu}$ are known?

In general, already the answer to the first question is negative. For example, we have by the Classical branching theorem,

$$\left(S^{(k,k)} + S^{(k+2,k-2)} + \dots\right) \downarrow_{\Sigma_{n-1}} = \left(S^{(k+1,k-1)} + S^{(k+3,k-3)} + \dots\right) \downarrow_{\Sigma_{n-1}},$$

(in the Grothendieck group of Σ_{n-1}) which shows that the answer to the first question is negative even in characteristic 0.

However, there is a special case, interesting for us, when both answers are positive. This is the case where all the composition factors of M belong to the same block, and this block is known (this assumption is of course satisfied if $M = S^\lambda$). The idea is that we can always find *one* composition factor, say D^μ, of M. Then we subtract the expression $\sum_\nu b_{\mu\nu} D^\nu = D^\mu \downarrow_{\Sigma_{n-1}}$ from $M \downarrow_{\Sigma_{n-1}}$ and repeat the procedure again.

To make this precise, we need the following lemma.

Lemma 3.4 *Let λ be a p-regular partition of n with*

$$\mathrm{cont}(\lambda) = (c_0, \dots, c_{p-1}).$$

Then for any composition factor D^ν of $D^\lambda \downarrow_{\Sigma_{n-1}}$ there exists a residue class $\alpha \in \{0, 1, \dots, p-1\}$ such that

$$\mathrm{cont}(\nu) = (c_0, \dots, c_{\alpha-1}, c_\alpha - 1, c_{\alpha+1}, \dots, c_{p-1}).$$

Proof. The composition factors of $D^\lambda \downarrow_{\Sigma_{n-1}}$ are among the composition factors of $S^\lambda \downarrow_{\Sigma_{n-1}}$. By the Classical branching theorem, D^ν is a composition factor of some $S^{\lambda_{A_j}}$, whence $\mathrm{cont}(\nu) = \mathrm{cont}(\lambda_{A_j})$, and the result follows. □

The following proposition provides us with the desired algorithm for determining the composition factors of a Σ_n-module M in a given block from the composition factors of $M \downarrow_{\Sigma_{n-1}}$, provided the branching coefficients for the restriction from Σ_n to Σ_{n-1} are known.

Proposition 3.5 *Assume that we have*

$$M = \sum_{\delta \vdash n} a_\delta D^\delta, \quad \textit{and} \quad M \downarrow_{\Sigma_{n-1}} = \sum_{\gamma \vdash n-1} c_\gamma D^\gamma.$$

(in the Grothendieck groups of Σ_n and Σ_{n-1}, respectively). Suppose also that $a_\delta \neq 0$ implies $\mathrm{cont}(\delta) = (c_0, \dots, c_{p-1})$ for some fixed numbers $c_0, \dots, c_{p-1}$.

Choose ν such that $c_\nu \neq 0$ but $\gamma \lhd \nu$ implies $c_\gamma = 0$. By Lemma 3.4, there exists a residue α modulo p such that

$$\mathrm{cont}(\nu) = (c_0, \dots, c_{\alpha-1}, c_\alpha - 1, c_{\alpha+1}, \dots, c_{p-1}).$$

Then

(i) *ν has at least one addable node of residue α;*
(ii) *if we define μ to be ν with the top addable node of residue α added, then μ is p-regular;*
(iii) *$a_\mu \neq 0$.*

Proof. Clearly D^ν is a composition factor of some $D^\mu \downarrow_{\Sigma_{n-1}}$ with $a_\mu \neq 0$. Moreover, by the choice of ν, D^ν is the bottom composition factor with respect to the dominance order. By Theorem 3.3, $\nu = \mu_A$ where A is the top removable node of μ such that μ_A is p-regular. This proves (i). Moreover, it follows from the definitions that A is the top addable node of μ_A of residue α, which proves (ii) and (iii). □

We now return to our historic remarks. In 1986, Benson [3] found the decomposition numbers for Σ_n for $p = 2$, $n = 14, 15$ (for $n \leq 13$ these were known [18]). He then used this new information to come up with five conjectures on the branching rules in characteristic 2. Even though these conjectures are now special cases of the theorems presented in Section 4 of this paper, we state the three main conjectures of Benson here. One of the reasons is that the "2-modular combinatorics" is much easier than the general "p-modular combinatorics", so the results in characteristic 2 have particularly simple form.

Benson's Conjectures. [3, Conjectures 1,4,5]. Let $p = 2$ and

$$\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_s > 0)$$

be a 2-regular partition of n.

(i) $D^\lambda \downarrow_{\Sigma_{n-1}}$ is irreducible if and only if

$$\lambda_1 \equiv \lambda_2 \equiv \dots \equiv \lambda_s \pmod 2,$$

in which case

$$D^\lambda \downarrow_{\Sigma_{n-1}} = D^{(\lambda_1 - 1, \lambda_2, \dots, \lambda_s)}.$$

(ii) $D^\lambda \downarrow_{\Sigma_{n-1}}$ is completely reducible if and only if

$$\lambda_i \equiv \lambda_{i+1} \pmod 2,$$

except possibly several even values of i, in which case

$$D^\lambda \downarrow_{\Sigma_{n-1}} = D^{(\lambda_1-1,\lambda_2,\dots,\lambda_s)} \oplus D^{(\lambda_1,\dots,\lambda_{j+1}-1,\dots,\lambda_s)}$$

where j is the minimal index such that $\lambda_j \not\equiv \lambda_{j+1}$.

(iii) $D^\lambda \uparrow^{\Sigma_{n+1}}$ is completely reducible if and only if

$$\lambda_1 \equiv \lambda_2 \equiv \dots \equiv \lambda_s \equiv 0 \pmod 2,$$

in which case

$$D^\lambda \uparrow^{\Sigma_{n+1}} = D^{(\lambda_1,\dots,\lambda_s,1)} \oplus D^{(\lambda_1,\lambda_2,\dots,\lambda_s+1)}.$$

Remark. Conjecture (ii) above is a corrected form of Conjecture 4 from [3]. The difference is in the set of composition factors which should be as here to make the conjecture true.

In 1992, Jantzen and Seitz [43, 23] have generalized Benson's conjecture (i) to the case of an arbitrary p. The generalization is not obvious. Gather together equal parts of λ to represent it in the form

$$\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_m^{a_m}), \quad l_1 > l_2 > \dots l_m > 0, \quad a_1, a_2, \dots, a_m > 0.$$

Conjecture of Jantzen-Seitz. [43, 23]. $D^\lambda \downarrow_{\Sigma_{n-1}}$ is irreducible if and only if

$$l_i - l_{i+1} + a_i + a_{i+1} \equiv 0 \pmod p$$

for all $i = 1, 2, \dots, m-1$, in which case

$$D^\lambda \downarrow_{\Sigma_{n-1}} = D^{\lambda_A}$$

where A is the top removable node of λ.

The "if-part" of the conjecture was proved in [23], as well as the "only-if-part" for $m \le 3$. The "only-if-part" in general was proved in [25, 10, 27]. We note that the papers [43] and [23] were motivated by the problem of describing the maximal subgroups in finite classical groups. The following more general question was also considered there: when is the restriction $D^\lambda \downarrow_{\Sigma_k \times \Sigma_{n-k}}$ irreducible? It is proved in [23] that $\dim D^\lambda > 1$ implies that such a restriction is always reducible for $1 < k < n-1$ and $p > 2$. In the case $p = 2$ this is not true, and as far as we know, the question remains open.

Finally, we mention a recent result of Sheth [44] who obtained a complete description of the branching rule for the two-part partitions, generalizing the dimension formulas of Erdmann [8]. Fix a two-part partition $\lambda = (n-m, m)$, and let

$$n - 2m = s_0 + s_1 p + \ldots + s_l p^l$$

be the p-adic expansion. Set t to be the minimal index such that $s_t < p-1$. In the following theorem we interpret $D^{(n-j,j)}$ as 0 if $j < 0$.

Theorem 3.6 [44]

(i) *Assume $p > 2$.*

- *If $n - 2m = 0$ then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}} = D^{(n-m,m-1)}$;*
- *If $n - 2m \equiv -2 \pmod p$ then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}} = D^{(n-m-1,m)}$;*
- *If $n - 2m \not\equiv -2, -1 \pmod p$ and $n - 2m > 0$ then*

$$D^{(n-m,m)} \downarrow_{\Sigma_{n-1}} = D^{(n-m-1,m)} \oplus D^{(n-m,m-1)};$$

- *If $n-2m \equiv -1 \pmod p$ then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}}$ is indecomposable with composition factors*

$$\begin{cases} D^{(n-m-1,m)}, \\ 2D^{(n-m-1+p^i,m-p^i)}, \; 0 \le i \le t-1, & \text{if } s_t = p-2; \\ D^{(n-m-1,m)}, \; D^{(n-m-1+p^t,m-p^t)}, \\ 2D^{(n-m-1+p^i,m-p^i)}, \; 0 \le i \le t-1, & \text{if } s_t < p-2. \end{cases}$$

(ii) *Assume $p = 2$.*

- *If n is even then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}} = D^{(n-m-1,m)}$;*
- *If n is odd then $D^{(n-m,m)} \downarrow_{\Sigma_{n-1}}$ is indecomposable with composition factors $D^{(n-m-1,m)}$, $2D^{(n-m-1+p^i,m-p^i)}$, $0 \le i \le t-1$.*

4. Various versions of branching rules

The general Branching problem formulated in Section 1 remains open. However, there has been a noticeable progress in the area during the last years. In this section we present an exposition of the main results of the papers [25, 27, 28, 29, 30, 32]. We use the notions introduced in Section 2.

Branching Theorems. *Let λ be a p-regular partition of n.*

(i) *For a removable node A of λ such that λ_A is p-regular, the irreducible module D^{λ_A} is a composition factor of $D^\lambda\downarrow_{\Sigma_{n-1}}$ if and only if A is normal. In this case the multiplicity is given by*

$$\left[D^\lambda\downarrow_{\Sigma_{n-1}}: D^{\lambda_A}\right] = \sharp\{\text{normal nodes } B \text{ above } A \text{ with } \mathrm{res} B = \mathrm{res} A\}.$$

(ii)

$$\mathrm{soc}(D^\lambda\downarrow_{\Sigma_{n-1}}) = \bigoplus_{A \text{ good}} D^{\lambda_A}.$$

(iii)

$$\begin{aligned}\dim_F \mathrm{End}_{\Sigma_{n-1}}(D^\lambda\downarrow_{\Sigma_{n-1}}) &= \dim_F \mathrm{End}_{\Sigma_{n+1}}(D^\lambda\uparrow^{\Sigma_{n+1}}) - 1 \\ &= \sharp\{\text{normal nodes of } \lambda\}.\end{aligned}$$

(iv) *For a p-regular partition μ of $n-1$ the vector space*

$$\mathrm{Hom}_{\Sigma_{n-1}}(S^\mu, D^\lambda\downarrow_{\Sigma_{n-1}})$$

is non-zero if and only if $\mu = \lambda_A$ for some normal node A of λ, in which case this space is 1-dimensional.

(v) *Let $A_1, \dots, A_d$ be all the good nodes of λ. Then*

$$D^\lambda\downarrow_{\Sigma_{n-1}} \cong I_1 \oplus \dots \oplus I_d$$

where

$$\mathrm{soc}\, I_j = \mathrm{head}\, I_j = D^{\lambda_{A_j}},\ j = 1, 2, \dots, d,$$

and the modules I_j are indecomposable, self-dual and all belong to distinct blocks. A similar result holds for $D^\lambda\uparrow^{\Sigma_{n+1}}$.
In particular,

$$\begin{aligned} &\sharp\{\text{indecomposable summands of } D^\lambda\downarrow_{\Sigma_{n-1}}\} \\ = \ &\sharp\{\text{block components of } D^\lambda\downarrow_{\Sigma_{n-1}}\} \\ = \ &\sharp\{\text{good nodes of } \lambda\},\end{aligned}$$

and

$$\begin{aligned} &\sharp\{\text{indecomposable summands of } D^\lambda\uparrow^{\Sigma_{n+1}}\} \\ = \ &\sharp\{\text{block components of } D^\lambda\uparrow^{\Sigma_{n+1}}\} \\ = \ &\sharp\{\text{good addable nodes for } \lambda\}.\end{aligned}$$

(vi) $D^\lambda \downarrow_{\Sigma_{n-1}}$ *is completely reducible if and only if all normal nodes of* λ *have distinct residues (equivalently, all normal nodes of* λ *are good), in which case*

$$D^\lambda \downarrow_{\Sigma_{n-1}} = \bigoplus_{A \text{ good}} D^{\lambda_A}.$$

(vii) $D^\lambda \uparrow^{\Sigma_{n+1}}$ *is completely reducible if and only if the number of normal nodes of* λ *is one less than the number of good addable nodes for* λ*, in which case*

$$D^\lambda \uparrow^{\Sigma_{n+1}} = \bigoplus_{A \text{ good addable}} D^{\lambda^A}.$$

(viii) $D^\lambda \downarrow_{\Sigma_{n-1}}$ *is irreducible if and only if* λ *has a unique normal node* A*, in which case* A *is the top removable node,* λ_A *is* p*-regular, and*

$$D^\lambda \downarrow_{\Sigma_{n-1}} = D^{\lambda_A}.$$

Remarks. (1) Note that for a normal node A, the partition λ_A may be not p-regular. However when we speak of the number of normal nodes, as in (i), (iii), (vii), we always take into account *all* normal nodes. On the other hand, one can easily check that λ_A is p-regular for any good node A, provided λ itself is p-regular. A similar result holds for good addable nodes. So (ii),(vi) and (vii) make sense.

(2) All branching theorems make sense in characteristic 0. This corresponds to the case "$p = \infty$". In particular, (i),(ii),(iv),(vi),(vii) are equivalent to the classical branching rule in this case; (vii) also claims that the number of the removable nodes is always one less than the number of the addable nodes.

(3) Part (iii) can be strongly generalized—it is possible to describe the algebra $\mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}})$ explicitly, see Theorem 4.1 below.

(4) Let $M^{(n-1,1)}$ denote the natural n-dimensional permutation module over Σ_n. Then

$$\begin{aligned}\mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}}) &\cong \mathrm{Hom}_{\Sigma_n}(D^\lambda \downarrow_{\Sigma_{n-1}} \uparrow^{\Sigma_n}, D^\lambda) \\ &\cong \mathrm{Hom}_{\Sigma_n}(D^\lambda \otimes M^{(n-1,1)}, D^\lambda) \cong \mathrm{Hom}_{\Sigma_n}(M^{(n-1,1)}, D^\lambda \otimes D^\lambda).\end{aligned}$$

So Branching theorem (iii) can be interpreted as a fact about tensor squares of irreducible modules.

(5) It is not difficult to verify that a p-regular partition has a unique normal node if and only if it satisfies the condition of Jantzen-Seitz as in Conjecture of Jantzen-Seitz in Section 3.

(6) Branching theorems (ii), (iv), and (vi) have been generalized to the Hecke algebras of type A by Brundan [5].

Now we detalize (iii). Let $A_1, A_2, \dots, A_d$ be the good nodes of λ. In view of (v), we may decompose

$$D^\lambda \downarrow_{\Sigma_{n-1}} = I_1 \oplus I_2 \oplus \dots \oplus I_d,$$

with

$$\operatorname{soc} I_j = \operatorname{head} I_j = D^{\lambda_{A_j}},$$

and the indecomposable modules $I_1, I_2, \dots, I_d$ belonging to the distinct blocks. Therefore

$$\operatorname{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}}) \cong \operatorname{End}_{\Sigma_{n-1}}(I_1) \oplus \operatorname{End}_{\Sigma_{n-1}}(I_2) \oplus \dots \oplus \operatorname{End}_{\Sigma_{n-1}}(I_d).$$

So we may concentrate on $\operatorname{End}_{\Sigma_{n-1}}(I_j)$, $1 \le j \le d$.

Theorem 4.1 [30]. *Let $1 \le j \le d$. Assume* $\operatorname{res} A_j = \alpha_j$, *and λ has m_j normal nodes of residue α_j. Then there exists an endomorphism*

$$\varphi_j \in \operatorname{End}_{\Sigma_{n-1}}(I_j)$$

such that $\varphi_j^{m_j} = 0$, and

$$\{\mathrm{id}_{I_j}, \varphi_j, \varphi_j^2, \dots, \varphi_j^{m_j - 1}\}$$

is a basis of $\operatorname{End}_{\Sigma_{n-1}}(I_j)$. *In particular, the algebra* $\operatorname{End}_{\Sigma_{n-1}}(I_j)$ *is isomorphic to the algebra of truncated polynomials $F[x]/(x^{m_j})$.*

Remarks. (1) Note that the irreducible module $D^{\lambda_{A_j}}$ which is isomorphic to the socle and the head of I_j appears in I_j exactly m_j times in view of Branching theorem (i). In this respect, the dimension of the endomorphism algebra of I_j behaves exactly like the dimension of the endomorphism algebra of the projective cover $P^{\lambda_{A_j}}$ of $D^{\lambda_{A_j}}$. If we represent I_j as a quotient $P^{\lambda_{A_j}}/Q$ of $P^{\lambda_{A_j}}$ then any endomorphism of $P^{\lambda_{A_j}}$ leaves Q invariant (this observation was made by Dipper and the author, compare with [7]).

(2) It follows from Theorem 4.1 and Branching theorem (i) that $I_j = D^{\lambda_{A_j}}$ if and only if $m_j = 1$.

We mention that tensoring with the sign representation and using the Mullineux map often gives an additional and very non-trivial information about the branching rule. The idea is as follows. Let m_n denote the Mullineux bijection (see Section 5). Then clearly

$$\left[D^{m_n(\lambda)} \downarrow_{\Sigma_{n-1}} : D^{m_n(\lambda)_A}\right] = \left[D^\lambda \downarrow_{\Sigma_{n-1}} : D^{m_{n-1}(m_n(\lambda)_A)}\right]. \qquad (2)$$

However, the LHS of (2) is known from Branching theorem (i).

We finish this section with an example.

Example. Let $\lambda = (8,4,4,3,1)$, $p = 3$. Then $D^\lambda\downarrow_{\Sigma_{19}} = I_1 \oplus I_2$ where $I_1 = D^{(8,4,4,3)}$, and the following information on I_2 is available.

- It is an indecomposable and self-dual module, whose endomorphism algebra is isomorphic to $F[x]/(x^2)$.
- The socle and the head of I_2 are isomorphic to $D^{(8,4,3,3,1)}$.
- The partitions corresponding to the composition factors of I_2 all have residue content $(6,6,7)$ and dominate $(7,4,4,3,1)$.
- $\left[I_2 : D^{(8,4,3,3,1)}\right] = 2$.
- $\left[I_2 : D^{(7,4,4,3,1)}\right] = 1$.
- $\left[I_2 : D^{(9,5,2,2,1)}\right] = 1$.

The last fact follows from the trick with the Mullineux map, see (2). We know of no other way to get this information.

5. An application to the Mullineux Conjecture

For an arbitrary irreducible $F\Sigma_n$-module D^λ, the tensor product $D^\lambda \otimes \mathrm{sgn}$ is an irreducible module again. So we can write

$$D^\lambda \otimes \mathrm{sgn} \cong D^{b_n(\lambda)}.$$

This defines an involutary bijection $b_n : P_n \to P_n$ on the set P_n of the p-regular partitions of n.

Mullineux problem. Describe b_n.

This problem is important by several reasons. One of them comes from the representation theory of the alternating group A_n (see e.g. [11]). If you try to describe the irreducible FA_n-modules using representations of Σ_n and the Clifford theory, you will face the Mullineux problem!

It is proved in [18, 8.15] that

$$S^\lambda \otimes \mathrm{sgn} \cong (S^{\lambda'})^* \tag{3}$$

where λ' is the partition conjugate to λ. In characteristic zero this provides a complete solution of our problem, since the Specht modules S^λ are the irreducible Σ_n-modules, and every irreducible module over Σ_n is self-dual. However in characteristic p the bijection b_n is much more difficult to describe.

The isomorphism (3) implies that $D^\lambda \otimes \mathrm{sgn}$ is a composition factor of $S^{\lambda'}$. So, using the Nakayama Conjecture, we obtain:

Let $\mathrm{cont}(\lambda) = (c_0, \dots, c_{p-1})$. Set $d_\alpha = c_{-\alpha},\ \alpha = 0, 1, \dots, p-1$.

$$\text{Then } \operatorname{cont}(b_n(\lambda)) = (d_0, \ldots, d_{p-1}) \tag{4}$$

Branching rules come into play via the following easy observation.

$$\operatorname{soc}\Big((D^\lambda \otimes \operatorname{sgn}) \downarrow_{\Sigma_{n-1}}\Big) \cong \Big(\operatorname{soc}(D^\lambda \downarrow_{\Sigma_{n-1}})\Big) \otimes \operatorname{sgn}. \tag{5}$$

It follows from (5) and Branching theorem (ii) that

$$\begin{gathered}\text{For any good node } A \text{ of } \lambda \text{ there exists a good} \\ \text{node } B \text{ of } b_n(\lambda) \text{ such that } b_{n-1}(\lambda_A) = b_n(\lambda)_B.\end{gathered} \tag{6}$$

We can say even more. If $\operatorname{res} A = \alpha$ then $\operatorname{res} B = -\alpha$. This follows immediately from (4).

The following combinatorial fact is not difficult to prove.

Lemma 5.1 [29, 1.4]. *Let μ be a p-regular partition and α be a fixed p-residue. Then μ has at most one good addable node of residue α.*

In view of Branching Theorem (ii), Lemma 5.1 has the following curious reformulation:

Corollary 5.2 [29, 3.3]. *If $\operatorname{soc}(D^\lambda \downarrow_{\Sigma_{n-1}})$ and $\operatorname{soc} D^\mu \downarrow_{\Sigma_{n-1}}$ have a common constituent then either $\lambda = \mu$ or $\operatorname{cont}(\lambda) \neq \operatorname{cont}(\mu)$.*

This surprising result shows that the properties (4) and (6) uniquely determine the bijections b_n. More precisely,

Theorem 5.3 [29, 4.2]. *Let $\{x_n : P_n \to P_n\}$, $n \geq 1$ be a family of maps satisfying the following two conditions:*

(1) for any $\lambda \in P_n$, $n \geq 1$, if $(c_0, \ldots, c_{p-1})$ is the content of λ then the content of $x_n(\lambda)$ is $(d_0, \ldots, d_{p-1})$ where $d_\alpha = c_{-\alpha}$ for $\alpha = 0, 1, \ldots, p-1$;

(2) for any $\lambda \in P_n$, $n \geq 2$, there exists a good node A of λ and a good node B of $x_n(\lambda)$ such that $x_{n-1}(\lambda_A) = x_n(\lambda)_B$.

Then $x_n = b_n$ for all $n \geq 1$.

This shows that we can describe $b_n : P_n \to P_n$ inductively as follows. Remove any good node A from λ. Assume $\operatorname{res} A = \alpha$. Then $b_n(\lambda)$ is the partition obtained from $b_{n-1}(\lambda_A)$ by adding a good node of residue $-\alpha$. Note that this is well defined in view of Lemma 5.1 and (6).

In practice we just remove good nodes one after another, until we come to the empty partition. For example, pick any good node A_1 of λ, then any good node A_2 of λ_{A_1}, and so on. If $\operatorname{res} A_j = \alpha_j$, we get a sequence $(\alpha_1, \ldots, \alpha_n)$ of p-residues. Then starting from the empty partition we build a new one by adding the good nodes of residues $-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1$, starting from $-\alpha_n$. It has been just proved that there *exists* exactly *one* way to do it.

In 1979, Mullineux [37] proposed another algorithm for calculating b_n. He defined, purely combinatorially, a bijection, say $m_n : P_n \to P_n$, and conjectured that $b_n = m_n$. This was called the Mullineux conjecture. (Another version of the Mullineux algorithm can be found in [46].) In calculations by hand the Mullineux algorithm works much faster than the algorithm described above. It has a number of other advantages. In [38], Mullineux has verified that m_n satisfies the property (1) of Theorem 5.3. Thus

Corollary 5.4 *The Mullineux Conjecture is true if and only if the following statement holds:*

(*) for any $\lambda \in P_n$, $n \geq 2$, there exists a good node A of λ and a good node B of $m_n(\lambda)$ such that $m_{n-1}(\lambda_A) = m_n(\lambda)_B$.

The combinatorial statement (*) turned out to be quite difficult to prove. The first proof was found by Ford and the author [12]. An easier but still quite intricate proof was later found by Bessenrodt and Olsson [4].

We finish this section by considering an important special case:

Example. Which p-regular partition corresponds to the sign representation? In other words, since sgn is irreducible we must have

$$\mathrm{sgn} \cong D^{\kappa}$$

for some p-regular partition κ. Clearly $\kappa = b_n((n))$, and using either algorithm one finds that

$$\kappa = ((d+1)^r, d^{p-1-r}) \quad \text{where} \quad n = d(p-1) + r,\ 0 \leq r < p-1.$$

Note that this fact also follows from an old paper of James [16]. Another proof, which uses the branching rules was found in [23, 4.3]. The fanciest proof known to the author uses the Schur functors and the fact that a special module of truncated polynomials over $GL_n(F)$ has the 1-dimensional 0-weight space.

6. An application to the ideals in the group algebras of the finitary symmetric and alternating groups

A permutation of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is called *finitary* if it fixes all but finitely many elements. The finitary symmetric group Σ_∞ is the group of all finitary permutations of $\mathbb{N}$. It can be represented as a union $\Sigma_\infty = \cup_{n\geq 1}\Sigma_n$. Similarly $A_\infty = \cup_{n\geq 1} A_n$ is the finitary alternating group. Clearly Σ_∞ and A_∞ are locally finite groups.

A. Zalesskii has shown (see [47] and the references there) that some ring theoretic questions on the group algebras of locally finite groups are closely related with the asymptotic behavior of the branching rules for finite groups. This fundamental observation accounts for the recent noticeable progress in the ideals theory of the group algebras of locally finite groups.

If a locally finite group G is countable (which is an unnecessary technical assumption) then it can be represented as a union, $G = \cup_{n\geq 1} G_n$, of its finite subgroups $G_1 \subset G_2 \subset \dots$. Given a (two-sided) ideal I in the group algebra FG we can form a family of ideals

$$I_n := I \cap FG_n \lhd FG_n, \; n = 1, 2, \dots.$$

This family has the property

$$I_n \cap FG_m = I_m, \quad \text{for any } 1 \leq m \leq n, \tag{7}$$

and I can be reconstructed from it as $I = \cup_{n\geq 1} I_n$. On the other hand, given a family of ideals $I_n \lhd FG_n$ satisfying the property (7) we may form a union $I = \cup_{n\geq 1} I_n$, and then $I \cap FG_n = I_n$ for any n. Thus the main problem is how to glue a big ideal from the small ones (equivalently, how to produce and classify the families of ideals satisfying (7)).

The right formalism for doing this comes from the representation theory of finite groups.

Definition (A. Zalesskii). Let Φ_n be a set of the isomorphism classes of irreducible FG_n-modules, $n = 1, 2, \dots$. The collection $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is called an *inductive system* (for G) if for any $m, n \in \mathbb{N}$ with $m < n$ the following two properties hold:

(1) For any $D \in \Phi_n$, all composition factors of the restriction $D\downarrow_{G_m}$ belong to Φ_m;

(2) For any $E \in \Phi_m$, there exists $D \in \Phi_n$ such that E is a composition factor of $D\downarrow_{G_m}$.

Theorem 6.1 [47, 8.1,1.25] (see also [2, 2.6]). *There is an order reversing bijection of partially ordered sets between the inductive systems and the semi-primitive ideals of FG.*

Since the maximal ideals of FG are clearly semi-primitive, we get

Corollary 6.2 *There is a 1-1 correspondence between the maximal ideals in FG and the minimal (non-empty) inductive systems for G.*

Now we specialize again to the case $G = \Sigma_\infty$ or A_∞. This special case is of particular importance by several reasons. One of them is the close connection between the ideals in $F\Sigma_\infty$ and the theory of PI-rings (see [1, 48, 40, 39, 14, 13]). A. Kemer has informed us that determining the prime ideals in $F\Sigma_\infty$ would be a crucial step in a classification of the prime varieties of associative algebras (cf [24]).

The ideal structure of $F\Sigma_\infty$ and FA_∞ is very rich. In case where $\mathrm{char}F = 0$ it was described in [39, 13]. Using Theorem 6.1 and the classical branching rule one can easily rederive such a description. On the other hand, the case of positive characteristic seems to be very difficult since the complete modular branching rule is not known.

Recently Baranov and the author [2] have described the *maximal* ideals in $F\Sigma_\infty$ and FA_∞, provided $p > 2$.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0)$ is a partition of n we define

$$h(\lambda) = m, \quad \chi(\lambda) = l_1 - l_m + m.$$

For $s = 1, 2, \ldots, p-1$ and $n > (p-s)(s-1)$ set

$$\Phi(s)_n = \{D^\mu \mid \mu \vdash n, \quad h(\mu) = s, \quad \chi(\mu) \leq p\}.$$

Theorem 6.3 [2]. *Let F be a field of characteristic $p > 2$.*

(i) *Set*

$$I(s)_n = \bigcap_{D \in \Phi(s)_n} \mathrm{Ann}_{F\Sigma_n}(D), \; n > (p-s)(s-1), \; s = 1, \ldots, p-1.$$

Then $I(s)_m = I(s)_n \cap F\Sigma_n$ for all $m < n$, $s = 1, 2, \ldots, p-1$. Moreover,

$$I(s) = \bigcup_{n > (p-s)(s-1)} I(s)_n, \quad s = 1, \ldots, p-1,$$

are exactly all distinct maximal (two-sided) ideals of $F\Sigma_\infty$, and

$$I(s) \cap F\Sigma_n = I(s)_n$$

for $n > (p-s)(s-1)$.

(ii) *Set*

$$J(t)_n = \bigcap_{D \in \Phi(t)_n} \mathrm{Ann}_{FA_n}(D{\downarrow}A_n), \; n > (p-t)(t-1), \; t = 1, \ldots, \frac{p-1}{2}.$$

Then $J(t)_m = J(t)_n \cap FA_n$ for all $m < n$, $t = 1, 2, \ldots, p-1$. Moreover,

$$J(t) = \bigcup_{n > (p-t)(t-1)} J(t)_n, \quad t = 1, \ldots, \frac{p-1}{2},$$

are exactly all distinct maximal (two-sided) ideals of FA_∞, and

$$J(t) \cap FA_n = J(t)_n$$

for $n > (p-t)(t-1)$.

(iii) $I(s) \cap FA_\infty = J(t)$ *where* $t = \min(s, p-s)$, $s = 1, \ldots, p-1$.

The main idea of the proof is to show that $\Phi(s) = \{\Phi(s)_n\}_{n \geq (p-s)(s-1)}$, $s = 1, 2, \ldots, p-1$ are exactly all distinct minimal inductive systems for Σ_∞, and to use Corollary 6.2. In doing so we had to use the branching rules in maximal available generality together with the Mullineux Conjecture!

Remark. Clearly

$$\mathrm{Aug}(F\Sigma_\infty) = \{\sum_{g \in \Sigma_\infty} \alpha_g g \mid \sum \alpha_g = 0\}$$

and

$$\mathrm{Aug}^\sigma(F\Sigma_\infty) = \{\sum_{g \in \Sigma_\infty} \alpha_g g \mid \sum \mathrm{sign}(g)\alpha_g = 0\}$$

are the maximal ideals in $F\Sigma_\infty$. These are $I(1)$ and $I(p-1)$, respectively. If $\mathrm{char} F = 0$ then $\mathrm{Aug}(F\Sigma_\infty)$ and $\mathrm{Aug}^\sigma(F\Sigma_\infty)$ are the only maximal ideals. In the modular case however we have $p-1$ maximal ideals, and $I(2), I(3), \ldots, I(p-2)$ have infinite codimensions.

The modules D^λ with $\chi(\lambda) \leq p$, which form the inductive systems $\Phi(s)$ have appeared before in various contexts, see [41, 35, 31].

Mathieu [35], using theory of tilting modules, found a combinatorial description of their dimensions. Let us consider the version of p-modular Young graph introduced in [29]. The vertices of this *directed* graph Y are all p-regular partitions, and two p-regular partitions μ and λ are connected by the arrow $\mu \to \lambda$ if and only if $\mu = \lambda_A$ for some normal node A. Mathieu proved that

$$\dim D^\lambda = \sharp\{\text{paths from } (1) \text{ to } \lambda \text{ in } Y\}, \text{ provided } \chi(\lambda) \leq p. \tag{8}$$

It is not difficult to deduce from the Branching theorems (see [31]) that $\chi(\lambda) \leq p$ implies that $D^\lambda \downarrow_{\Sigma_{n-1}}$ is completely reducible, and every constituent D^μ of this restriction also has the property $\chi(\mu) \leq p$. This yields a simple inductive proof of (8), and also shows that when considering the dimensions of the modules D^λ with $\chi(\lambda) \leq p$ we can restrict ourselves to the paths in the complete subgraph of Y whose vertices are the partitions ν with $\chi(\nu) \leq p$. This is how the result is formulated in [35].

A second version of a p-modular Young graph has the same vertices but fewer arrows: we connect $\mu \to \lambda$ if and only if $\mu = \lambda_A$ for some *good* node A. This version plays a fundamental role in the recently discovered LLT-algorithm [34].

We define a third version of a p-modular Young graph. The vertices of this directed graph Y' are, as usual, all p-regular partitions, but Y' has much more arrows. We draw i arrows from μ to λ if $\mu = \lambda_A$ for some normal node A such that there are $(i-1)$ normal nodes B above A with res$B$ = res$A$ *or* if $\lambda = m_{n-1}(m_n(\lambda)_C)$ for some normal node C of $m_n(\lambda)$ such that there are $(i-1)$ normal nodes D above C with res$D$ = res$C$. (Recall that we denote by m_n the Mullineux map, see Section 5). Now it follows immediately from Branching theorem (i) and the Mullineux conjecture that

$$\dim D^\lambda \geq \sharp\{\text{paths from } (1) \text{ to } \lambda \text{ in } Y'\}, \text{ for any } p\text{-regular } \lambda. \qquad (9)$$

7. Some related problems on algebraic groups

In this section we assume that the basic field F is algebraically closed.

The proofs of the Branching theorems presented in Section 4 are obtained in two steps. First, the Schur functors [15, 23] are used to translate the problem into a problem on $GL_n(F)$ (or $SL_n(F)$), and then this latter problem is solved using the algebraic groups methods. Both steps have rather subtle points. For example, to ensure that Schur functors "work properly", we need the following result, interesting in its own right.

Theorem 7.1 [25]. *Let $L(\lambda)$ be the irreducible (rational) $SL_n(F)$-module with p-restricted highest weight λ. Then every irreducible constituent of*

$$\operatorname{soc} L(\lambda)\downarrow_{SL_{n-1}(F)} \cong \operatorname{head} L(\lambda)\downarrow_{SL_{n-1}(F)}$$

has a p-restricted highest weight.

This fact is rather surprising since in general the restricted module $L(\lambda)\downarrow_{SL_{n-1}(F)}$ has a lot of non-p-restricted composition factors. Theorem 7.1 can be developed as follows.

Theorem 7.2 [26]. *Let λ be an arbitrary dominant weight, and*

$$L(\lambda) = L(\lambda^{(0)}) \otimes L(\lambda^{(1)})^{[1]} \otimes \ldots \otimes L(\lambda^{(d)})^{[d]}$$

be the presentation of $L(\lambda)$ as the Steinberg tensor product, with every $\lambda^{(d)}$ p-restricted. Then

$$\begin{aligned} \operatorname{soc} L(\lambda)\downarrow_{SL_{n-1}(F)} &\cong \operatorname{soc} L(\lambda^{(0)})\downarrow_{SL_{n-1}(F)} \\ &\otimes \left(\operatorname{soc} L(\lambda^{(1)})\downarrow_{SL_{n-1}(F)}\right)^{[1]} \otimes \ldots \otimes \left(\operatorname{soc} L(\lambda^{(d)})\downarrow_{SL_{n-1}(F)}\right)^{[d]}. \end{aligned} \qquad (10)$$

Remarks. (1) Note that the tensor product in (10) is completely reducible in view of Theorem 7.1. We can reformulate the statement of Theorem 7.2 as follows: the irreducible constituents of $\operatorname{soc} L(\lambda)\downarrow_{SL_{n-1}(F)}$ are all the modules of the form

$$L(\mu^{(0)}) \otimes L(\mu^{(1)})^{[1]} \otimes \ldots \otimes L(\mu^{(d)})^{[d]}$$

where $\mu^{(j)}$ runs over the highest weights of all irreducible components of the module $\operatorname{soc} L(\lambda^{(j)}) \downarrow_{SL_{n-1}(F)}$, $j = 0, 1, \ldots, d$.

(2) Theorem 7.1 can be generalized to the embeddings of algebraic groups of types $C_{l-1} < C_l$, $D_l < B_l$, $B_l < D_{l+1}$. On the other hand, if we restrict ourselves to the Levi-type subgroups then Theorem 7.1 is *only* true for the embeddings $A_{l-1} < A_l$ and $C_{l-1} < C_l$. On the other hand Theorem 7.2 is true for the embeddings $A_{l-1} < A_l$, $D_l < B_l$, $B_l < D_{l+1}$. We refer the reader to [26] for more details.

(3) Theorem 7.1 is true not only for $L(\lambda)$ but also for any submodule of the co-Weyl module $\nabla(\lambda)$, in particular for $\nabla(\lambda)$ itself. (This was observed by Brundan and the author).

Now we consider a problem which, modulo some intricate combinatorics [32], is essentially equivalent to Branching theorem (i). The problem is to describe the dimension of the weight space $L(\lambda)_{\lambda-\alpha}$ over $SL_n(F)$ for any positive root α.

We introduce some notation. It is convenient to work with the group $SL_{l+1}(F)$ considered as an algebraic group of type A_l. Let

$$\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$$

be a base in the root system A_l, and

$$\{\omega_1, \omega_2, \ldots, \omega_l\}$$

be the corresponding fundamental weights. Any positive root in A_l has a form

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_j \quad \text{with } 1 \leq i \leq j \leq l.$$

One can use Smith's theorem [45] to reduce our problem to the case where

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l$$

is the highest root (otherwise we just "live" in a smaller SL_n). Let

$$\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \ldots + \lambda_l\omega_l.$$

Those λ_i which are non-zero modulo p play a particular role in our problem. So put

$$\{i_1 < i_2 < \ldots < i_k\} = \{i \mid 1 \leq i \leq l, \lambda_i \not\equiv 0\}.$$

The difficulty of our problem "grows quickly" with the growth of k. Let us denote

$$d(\lambda) = \dim L(\lambda)_{\lambda-\alpha_1-\lambda_2-\ldots-\alpha_l};$$

$$x_j = i_j - i_{j-1},\ j = 2, 3, \dots, k;$$
$$B_{j,m} = \sum_{q=j}^{m} \lambda_{i_q} + i_m - i_j,\ 1 \le j \le m \le k.$$

Then (writing "$\equiv$" for "$\equiv \pmod p$") we have:

(i) If $k = 0$ then $d(\lambda) = 0$ (clear).

(ii) If $k = 1$ then $d(\lambda) = 1$ (clear).

(iii) Let $k = 2$. Then

$$d(\lambda) = \begin{cases} x_2, & \text{if } \mathcal{B}_{1,2} \equiv 0, \\ x_2 + 1, & \text{otherwise.} \end{cases}$$

(this is an easy calculation, see [42, 8.6])

(iv) Let $k = 3$. Then

$$d(\lambda) = \begin{cases} x_2 x_3 + 1, & \text{if } \mathcal{B}_{1,2} \equiv 0,\, \mathcal{B}_{2,3} \equiv 0; \\ x_2(x_3 + 1) + 1, & \text{if } \mathcal{B}_{1,2} \equiv 0,\, \mathcal{B}_{2,3} \not\equiv 0; \\ (x_2 + 1)x_3 + 1, & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\, \mathcal{B}_{2,3} \equiv 0; \\ (x_2 + 1)(x_3 + 1) - 1, & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\, \mathcal{B}_{2,3} \not\equiv 0,\, \mathcal{B}_{1,3} \equiv 0; \\ (x_2 + 1)(x_3 + 1), & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\, \mathcal{B}_{2,3} \not\equiv 0,\, \mathcal{B}_{1,3} \not\equiv 0. \end{cases}$$

(this is a difficult calculation)

(v) Let $k = 4$. Then the value of $d(\lambda)$ can be found from Table 1.

Part (v) is very difficult to get by elementary calculations. We have deduced it from the general result which we now present.

The following definition resembles the definition of normal nodes in Young diagrams. This is not a coincidence.

Definition 7.3 Let $1 \le j \le m \le k$. We want to define a number $\sigma_{j,m}$. This number is 0 or 1 depending on whether a certain combinatorial condition holds:

if for every q, $j < q \le m$, with $B_{j,q} \equiv 0 \pmod p$ there exists $t(q)$, $j < t(q) < q$ such that $B_{t(q),q} \equiv 0 \pmod p$ and such that $q \ne q'$ implies $t(q) \ne t(q')$, we put $\sigma_{j,m} = 1$.

otherwise we put $\sigma_{j,m} = 0$.

Now we define a $k \times k$ matrix $M(\lambda) = \|a_{tu}\|_{1 \le t,u \le k}$ by setting

$$a_{t,u} = \begin{cases} \sigma_{t,u-1} + (i_{u+1} - i_u - 1)\sigma_{t,u}, & \text{if } 1 \le t < u < k; \\ \sigma_{tk}, & \text{if } u = k; \\ -1, & \text{if } u = t - 1; \\ 0, & \text{otherwise.} \end{cases}$$

TABLE 1.

$d(\lambda)$	$\mathcal{B}_{1,2}$	$\mathcal{B}_{1,3}$	$\mathcal{B}_{1,4}$	$\mathcal{B}_{2,3}$	$\mathcal{B}_{3,4}$	$\mathcal{B}_{2,4}$
$x_2x_3x_4+x_2+x_4$	$\equiv 0$	any	any	$\equiv 0$	$\equiv 0$	any
$(x_2+1)x_3x_4+x_2+x_4+1$	$\not\equiv 0$	any	any	$\equiv 0$	$\equiv 0$	any
$x_2x_3(x_4+1)+x_2+x_4+1$	$\equiv 0$	any	any	$\equiv 0$	$\not\equiv 0$	any
$(x_2+1)x_3(x_4+1)+x_2+x_4+1$	$\not\equiv 0$	any	$\not\equiv 0$	$\equiv 0$	$\not\equiv 0$	any
as above	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\equiv 0$
$(x_2+1)x_3(x_4+1)+x_2+x_4$	$\not\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\not\equiv 0$	any
as above	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$x_2(x_3+1)x_4+x_2+x_4$	$\equiv 0$	any	any	$\not\equiv 0$	$\equiv 0$	any
$x_2(x_3+1)x_4+x_2+x_3x_4+1$	$\not\equiv 0$	$\equiv 0$	any	$\not\equiv 0$	$\equiv 0$	any
$(x_2+1)(x_3+1)x_4+x_2+1$	$\not\equiv 0$	$\not\equiv 0$	any	$\not\equiv 0$	$\equiv 0$	any
$x_2x_3(x_4+1)+(x_2+1)x_4+1$	$\equiv 0$	any	any	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$
$x_2(x_3+1)x_4+x_2x_3+x_3(x_4+1)+1$	$\not\equiv 0$	$\equiv 0$	any	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$
$(x_2+1)x_3(x_4+1)+(x_2+1)x_4+1$	$\not\equiv 0$	$\not\equiv 0$	any	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$
$x_2(x_3+1)(x_4+1)+(x_4+1)$	$\equiv 0$	any	any	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$x_2(x_3+1)(x_4+1)+x_3(x_4+1)+1$	$\not\equiv 0$	$\equiv 0$	any	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$x_2(x_3+1)(x_4+1)+x_3(x_4+1)+x_4$	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$(x_2+1)(x_3+1)(x_4+1)$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$

For example if $k=4$ we get

$$M(\lambda)=\begin{bmatrix} x_2 & \sigma_{1,1}+(x_3-1)\sigma_{1,2} & \sigma_{1,2}+(x_4-1)\sigma_{1,3} & \sigma_{1,4} \\ -1 & x_3 & \sigma_{2,2}+(x_4-1)\sigma_{2,3} & \sigma_{2,4} \\ 0 & -1 & x_4 & \sigma_{3,4} \\ 0 & 0 & -1 & \sigma_{4,4} \end{bmatrix}.$$

Now we can formulate the following important result

Theorem 7.4 [32, 1.17] $\dim L(\lambda)_{\lambda-\alpha_1-\alpha_2-\ldots-\alpha_l}=\det M(\lambda)$.

Remarks. (1) Cofactor expansion of $\det M(\lambda)$ along the first row yields a recurrent formula for $d(\lambda)$.

(2) Theorem 7.4 is just one of the different descriptions of $d(\lambda)$ found in [32]. There we also present various bases for the weight space

$$L(\lambda)_{\lambda-\alpha_1-\alpha_2-\ldots-\alpha_l}.$$

Now consider the following, more special question. Assume the positions $i_1 < i_2 < \ldots < i_k$ for which $\lambda_{i_j} \not\equiv 0 \pmod p$ are fixed. What is the minimal possible value for $d(\lambda)$ and for which λ this minimal value takes place?

Definition 7.5 We define the integers $f(i_j, i_{j+1}, \ldots, i_k)$, $j = k, k-1, \ldots, 1$, by setting

$$\begin{aligned} f(i_k) &= 1; \\ f(i_{k-1}, i_k) &= i_k - i_{k-1}; \\ f(i_j, i_{j+1}, \ldots, i_k) &= (i_{j+1} - i_j) f(i_{j+1}, i_{j+2}, \ldots, i_k) \\ &\quad + f(i_{j+2}, i_{j+3}, \ldots, i_k), \\ &\quad j = k-2, k-3, \ldots, 1. \end{aligned}$$

Example. Let $i_1 = 1,\ i_2 = 2, \ldots, i_l = l$. Then $f(1, 2, \ldots, l) = f_l$ where f_j denotes the j-th Fibonacci number defined by $f_1 = 1,\ f_2 = 1$ and $f_j = f_{j-1} + f_{j-2}$ for $j \geq 3$.

Theorem 7.6 [32, 1.21] *Assume that*

$$\{i_1 < i_2 < \ldots < i_k\} = \{i \mid 1 \leq i \leq l,\ \lambda_i \not\equiv 0 \pmod p\} \neq \emptyset.$$

Then

$$\dim L(\lambda)_{\lambda-\alpha_1-\alpha_2-\ldots-\alpha_l} \geq f(i_1, i_2, \ldots, i_k).$$

Moreover,

$$\dim L(\lambda)_{\lambda-\alpha_1-\alpha_2-\ldots-\alpha_l} = f(i_1, i_2, \ldots, i_k).$$

if and only if

$$B_{j,j+1} \equiv 0 \pmod p \quad \textit{for all } j = 1, 2, \ldots, k-1.$$

Dominant weights $\lambda = \lambda_{i_1}\omega_1 + \lambda_{i_2}\omega_2 + \ldots + \lambda_{i_k}\omega_k$ with $\lambda_{i_j} \not\equiv 0 \pmod p$ for $j = 1, 2, \ldots, k$ and $B_{j,j+1} \equiv 0 \pmod p$ for $j = 1, 2, \ldots, k-1$, have first appeared in the paper [23]. Since then they showed up in various contexts. They also play a role in a special branching rule for general linear groups which we now describe.

It is natural to expect that by analogy with symmetric groups, one might be able to obtain certain branching theorems for general linear groups.

However this problem turns out to be much more difficult. We believe that the first problem which should be considered is a description of the socle of the restriction, $L\downarrow_{GL_{n-1}(F)}$, of an irreducible $GL_n(F)$-module L to the subgroup $GL_{n-1}(F)$. This problem is unsolved at the moment. The best result available now is due to Brundan, the author and I. Suprunenko [6]. In this paper we get a criterion for the complete reducibility of $L\downarrow_{GL_{n-1}(F)}$, and in the case where the restriction $L\downarrow_{GL_{n-1}(F)}$ *is* completely reducible we describe its composition factors. To explain the results obtained it is convenient to return to the partition notation.

If λ is a partition of a non-negative integer r with at most n parts we can identify it with a dominant weight for $GL_n(F)$. The (polynomial) irreducible $GL_n(F)$-module with highest weight λ will be denoted by $L(\lambda)$ or $L_n(\lambda)$.

Using Theorems 7.1 and 7.2 we can reduce all the questions under consideration to the case where λ is p-restricted (i.e. $\lambda_i - \lambda_{i+1} < p$ for $i = 1, 2, \dots, n-1$). More precisely, these theorems imply that a module

$$L(\lambda) = L(\lambda^{(0)}) \otimes L(\lambda^{(1)})^{[1]} \otimes \dots \otimes L(\lambda^{(d)})^{[d]}$$

has a completely reducible restriction to $GL_{n-1}(F)$ if and only if every module $L(\lambda^{(j)})$, $j = 1, 2, \dots, d$, does. Moreover, in this case Theorem 7.2 tells us how to find the composition factors of the restriction $L(\lambda)\downarrow_{GL_{n-1}(F)}$ if the composition factors of the restrictions $L(\lambda^{(j)})\downarrow_{GL_{n-1}(F)}$, $j = 1, 2, \dots, d$, are known.

So from now on we assume that λ is p-restricted.

Let us give the necessary combinatorial definitions. We fix a p-restricted dominant weight λ (i.e. $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$, and $\lambda_i - \lambda_{i-1} < p$ for $i = 1, 2, \dots, n-1$). Let

$$\{i_1 < i_2 < \dots < i_k\} = \{i \mid 1 \leq i < n,\ \lambda_i > \lambda_{i+1}\}. \tag{11}$$

Put

$$B^{jm} := i_m - i_j + \lambda_{i_j} - \lambda_{i_m+1},\ 1 \leq j \leq m \leq k, \tag{12}$$
$$R_j := \{\mathrm{res}(i_j, l) \mid \lambda_{i_j+1} < l \leq \lambda_{i_j}\},\ j = 1, 2, \dots, k. \tag{13}$$

Then B^{jm} is the length of a special hook, and R_j is the set of the residues of such nodes in the row i_j which are the bottoms of their columns.

We say that λ is *GJS* (generalized Jantzen-Seitz) if *at least one* of the following two conditions holds:

(1) $B^{j,j+1} \equiv 0 \pmod p$ for $j = 1, 2, \dots, k-1$;

(2) $R_j \cap R_m = \emptyset$ as long as $1 \leq j \neq m \leq k$.

Let $\mu = (\mu_1 \geq \ldots \geq \mu_{n-1} \geq 0)$ be a $GL_{n-1}(F)$-weight. We say μ is *allowable* (for λ) if the following two conditions hold:

(a)$\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for all $1 \leq i < n$;

(b) if $\mu_{i_j} < \lambda_{i_j}$ and $B^{jm} \equiv 0 \pmod p$ for some $m > j$ ($k \geq m > j \geq 1$) then $\mu_{i_m} = \lambda_{i_m+1}$.

Theorem 7.7 [6] *Let $L_n(\lambda)$ be an irreducible (polynomial) $GL_n(F)$-module with p-restricted highest weight λ. Then the restriction of $L_n(\lambda)$ to the natural subgroup $GL_{n-1}(F)$ is completely reducible if and only if λ is GJS, in which case*

$$L_n(\lambda)\downarrow_{GL_{n-1}(F)} \cong \bigoplus_{\mu \text{ allowable}} L_{n-1}(\mu).$$

Now we make a much stronger assumption: we want $L_n(\lambda)$ to be completely reducible under the restrictions to any Levi subgroups of $GL_n(F)$. The representation $L_n(\lambda)$ satisfying this assumption will be called *completely splittable*. The analogous notion for the symmetric groups was introduced in [31]. It was proved there that an irreducible $F\Sigma_n$-module D^λ is completely splittable (i.e. is completely reducible under the restrictions to all Young subgroups) if and only if $\chi(\lambda) \leq p$, see Section 6. Surprisingly, we get essentially the same answer for the general linear groups (up to the conjugation and modulo Steinberg's tensor products).

First of all note that using Theorems 7.1 and 7.2 as above one can show that a module

$$L(\lambda) = L(\lambda^{(0)}) \otimes L(\lambda^{(1)})^{[1]} \otimes \ldots \otimes L(\lambda^{(d)})^{[d]}$$

is completely splittable if and only if every module $L(\lambda^{(j)})$, $j = 1, 2, \ldots, d$, is. So we may assume that λ is p-restricted.

Set

$$\psi(\lambda) = B^{1k},$$

see (11) and (12).

Theorem 7.8 [6] *An irreducible $GL_n(F)$-module $L_n(\lambda)$ with p-restricted highest weight λ is completely splittable if and only if $\psi(\lambda) \leq p$.*

Example. One can use Theorem 7.8 and Steinberg's tensor product to show that $L(a\omega_j)$ is a completely splittable $SL_n(F)$-module for any $a \in \mathbb{Z}_{\geq 0}$ and $1 \leq j < n$.

Completely splittable representations are of particular importance because they admit an explicit description of their formal characters. The reason is that not only the restriction of such a representation to GL_{n-1} is completely reducible but every composition factor of this restriction is completely splittable again. So we may apply induction and the *known* branching rule. This leads to the following result.

Let λ be p-restricted, and $\psi(\lambda) \leq p$. We define a *p-hook* to be a pair of nodes (i, j) and (i', j') in λ with $i - j + p - 1 = i' - j'$. We refer to (i, j) as the top-right node in the p-hook, and (i', j') as the bottom-left node. Given a standard λ-tableau t, we say a p-hook is *bad* (in t) if

(1) the entry in the top-right node of the hook is strictly larger than the entry in the bottom-left node;

(2) no entry in the column containing the bottom-left node is equal to the entry in the top-right node.

Remark. We note that the top-right node of a p-hook may be to the left of its bottom-left node! However this never happens for bad p-hooks, which justifies the terminology.

Theorem 7.9 [6] *Let λ be p-restricted, and $\psi(\lambda) \leq p$. Then*

$$\dim L(\lambda)_\mu = \sharp\{\text{standard } \lambda\text{-tableaux of type } \mu \text{ without bad } p\text{-hooks}\}.$$

We mention that an equivalent character formula was obtained independently by Mathieu and Papadopoulo [36] using a tilting modules approach.

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