# ON DECOMPOSITION NUMBERS AND BRANCHING COEFFICIENTS FOR SYMMETRIC AND SPECIAL LINEAR GROUPS

ALEXANDER KLESHCHEV

## Introduction

In [13] – [15] we have obtained results generalizing the classical Branching Theorem for symmetric groups to the case of modular representations. For example we have described the socle of the restriction, $D^\lambda \downarrow_{\Sigma_{n-1}}$, of an arbitrary (modular) irreducible representation $D^\lambda$ of the symmetric group $\Sigma_n$ to the natural subgroup $\Sigma_{n-1}$. In particular, it turns out that this socle is always multiplicity-free. However, the multiplicity of a composition factor in $D^\lambda \downarrow_{\Sigma_{n-1}}$ does not need to be equal to 1. This is one of the phenomena which make the modular case very complicated in comparison with the case of ordinary representations. Moreover, it has been proved by James [7, Theorem C] that a composition multiplicity in $D^\lambda \downarrow_{\Sigma_{n-1}}$ can take *any* value (a similar result has been obtained by Jantzen and Seitz [12, 3.3]). One of the main results of this paper describes the multiplicities of *some* composition factors in $D^\lambda \downarrow_{\Sigma_{n-1}}$. It contains the mentioned results of James and Jantzen-Seitz as a special case.

More precisely, let $K$ be a field of characteristic $p > 0$ and let

$$\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0) \tag{1}$$

be a partition of $n$. If $1 \leq j \leq m$ and $l_j > l_{j+1}$ ($l_{m+1}$ is interpreted as 0) we define a partition $\lambda(j)$ of $n-1$ as follows:

$$\lambda(j) = (l_1, \ldots, l_{j-1}, l_j - 1, l_{j+1}, \ldots, l_m). \tag{2}$$

The Young diagram of $\lambda(j)$ is obtained from that of $\lambda$ by removing the last box in the $j$-th row. (Note that $\lambda(j)$ meant a different thing in [14].) If $\lambda$ is $p$-regular we denote by $D^\lambda$ the corresponding irreducible $K\Sigma_n$-module.

We describe the multiplicity

$$\left[D^\lambda \downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}\right]$$

for any $j$ such that $l_j > l_{j+1}$ and $\lambda(j)$ is $p$-regular (i.e. for all cases where $D^{\lambda(j)}$ makes sense).

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Unlike "the case of zero characteristic", in general there can be composition factors of $D^\lambda\downarrow_{\Sigma_{n-1}}$ which are different from any $D^{\lambda(j)}$. However we can obtain more information about them, too, using the following trick. Let $\mathrm{sgn}_n$ be the sign representation of $\Sigma_n$. Then $D^\lambda\otimes\mathrm{sgn}_n$ is the irreducible module $D^{b_n(\lambda)}$ corresponding to a ($p$-regular) partition $b_n(\lambda)$ described in [15], [5] and [1]. We clearly have

$$\left[D^\lambda\downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}\right] = \left[(D^\lambda\otimes\mathrm{sgn}_n)\downarrow_{\Sigma_{n-1}} : (D^{\lambda(j)}\otimes\mathrm{sgn}_{n-1})\right].$$

So if $b_n(\lambda)=(\mu_1\geq\mu_2\geq\dots\geq\mu_r>0)$, $1\leq k\leq r$ and $\mu_k>\mu_{k+1}$ then we also know the multiplicity

$$\left[D^\lambda\downarrow_{\Sigma_{n-1}} : D^{b_{n-1}((b_n(\lambda))(k))}\right].$$

This often provides a lot of a new information since $b_{n-1}\Big((b_n(\lambda))(k)\Big)$ is not generically of the form $\lambda(j)$, cf. Example after Theorem 1.4. We note that in general there can be composition factors of $D^\lambda\downarrow_{\Sigma_{n-1}}$ different from any $\lambda(j)$ and $b_{n-1}\Big((b_n(\lambda))(k)\Big)$, cf. the same example and [25].

The new information on the composition factors of $D^\lambda\downarrow_{\Sigma_{n-1}}$ and their multiplicities allows us to improve the lower bound for $\dim D^\lambda$ obtained in [15] since we now consider multiple edges in the modular Young graph (and add some new edges).

For a partition $\lambda$ of $n$ written in the form (1) and integers $i,j$ such that $1\leq i<j\leq m$, $l_i<l_{i-1}$, $l_j>l_{j+1}$ ($l_0$ and $l_{m+1}$ are interpreted as $+\infty$ and 0, respectively) we define a partition $\lambda(i,j)$ of $n$ as follows:

$$\lambda(i,j)=(l_1,\dots,l_{i-1},l_i+1,l_{i+1},\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_m). \tag{3}$$

The Young diagram of $\lambda(i,j)$ is obtained from that of $\lambda$ by moving the last box in the $j$-th row of $\lambda$ to the end of its $i$-th row.

We denote by $S^\lambda$ the Specht module corresponding to $\lambda$. The second main result of the paper is a description of the decomposition numbers

$$d_{\lambda,\lambda(i,j)}=\left[S^\lambda : D^{\lambda(i,j)}\right].$$

Of course, we need to assume here that $\lambda(i,j)$ is $p$-regular. Note that though $\dim\mathrm{Hom}_{\Sigma_n}(S^{\lambda(i,j)},S^\lambda)\leq 1$ (this is a special case of the main result of [3]), decomposition numbers $d_{\lambda,\lambda(i,j)}$ can take *any* values.

Again, using a trick with tensoring by the sign representation, we can obtain additional information on the decomposition numbers. Indeed, we have

$$\left[S^\lambda : D^{\lambda(i,j)}\right] = \left[S^\lambda\otimes\mathrm{sgn}_n : D^{\lambda(i,j)}\otimes\mathrm{sgn}_n\right].$$

So, in view of the isomorphism $S^\lambda\otimes\mathrm{sgn}_n\cong(S^{\lambda'})^*$, we also know all decomposition numbers of the form

$$\left[S^\lambda : D^{b_n(\lambda'(k,l))}\right]$$

where $b_n$ is the Mullineux bijection and $\lambda'$ is the partition conjugate to $\lambda$.

We also compute the multiplicities

$$k_{\lambda,\lambda(i,j)} = \left[M^\lambda : Y^{\lambda(i,j)}\right]$$

of the Young modules $Y^{\lambda(i,j)}$ in the permutation modules $M^\lambda$ for arbitrary $\lambda$, $i$ and $j$. We note that the constants $k_{\lambda,\mu}$ are modular analogues of the Kostka numbers. By Klyachko's Theorem [17] (see also the proof of Theorem 4.6.3 in [18]) they can also be identified with the weight multiplicities of the irreducible polynomial $GL_n(K)$-representations.

Actually, in this paper we first work with the algebraic group $SL_n(K)$. Then we translate the results obtained to the language of symmetric groups using Schur functors and Klyachko's Theorem.

The results on the algebraic groups (of type $A$) obtained might have an independent interest so we mention here some of them. Let $\lambda$ be a dominant weight. We denote by $V(\lambda)$ and $L(\lambda)$ the Weyl module and the irreducible module with highest weight $\lambda$ over $SL_n(K)$, respectively. For an arbitrary root $\alpha$ we find the weight multiplicity

$$\dim L(\lambda)_{\lambda-\alpha}$$

and the composition multiplicity

$$[V(\lambda) : L(\lambda-\alpha)]\,.$$

In particular, it turns out that $[V(\lambda) : L(\lambda-\alpha)]$ can take *any* value though, by [3], the dimension of the space $\mathrm{Hom}_{SL_n(K)}(V(\lambda-\alpha), V(\lambda))$ is at most 1. Moreover we describe some bases of the weight space $L(\lambda)_{\lambda-\alpha}$, see Theorems 8.6, 8.8 and 8.9. We also investigate the question when the weight multiplicity $\dim L(\lambda)_{\lambda-\alpha}$ is minimal, provided non-zero components of $\lambda$ are fixed. It turns out that for a weight in general position the minimal possible value for this multiplicity is given by a *Fibonacci number.*

Exact formulations of the main results are given in Section 1. The rest of the paper is organized as follows.

- Part I. *Preliminary combinatorial results.*
  - Section 2. *Essential subsets.*
  - Section 3. *Normality.*
  - Section 4. *$\sigma$-subsets.*
  - Section 5. *Recurrence formulas for the number of essential subsets.*
  - Section 6. *Quasi-latticed sequences and sets.*
- Part II. *Main Results.*
  - Section 7. *Preliminaries from representation theory.*
  - Section 8. *Dimension and bases of the weight space $L(\lambda)_{\lambda-\alpha}$.*
  - Section 9. *Some composition multiplicities of $L(\lambda)_1|A_{l-1}(K)$.*
  - Section 10. *Decomposition numbers $[V(\lambda) : L(\lambda-\alpha)]$.*
  - Section 11. *How to translate from $SL_n$ to $\Sigma_n$.*

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## 1. Formulations of the main results

Throughout the paper:

$K$ is a field of characteristic $p > 0$. All modules are supposed to be vector spaces over $K$.

$\mathbb{Z}_+$ is the set of non-negative integers.

For integers $r, s$ we write $r \equiv s$ for $r \equiv s \pmod p$ and denote

$$[r,s] = \{i \in \mathbb{Z} \mid r \leq i \leq s\}, \qquad (r,s] = \{i \in \mathbb{Z} \mid r < i \leq s\},$$
$$[r,s) = \{i \in \mathbb{Z} \mid r \leq i < s\}, \qquad (r,s) = \{i \in \mathbb{Z} \mid r < i < s\}.$$

We consider a (fixed) partition $\lambda$ of $n$ written in the form (1). We always interpret $l_0$ as $+\infty$ and $l_{m+1}$ as 0.

Definition of the following four series of $K\Sigma_n$-modules can be found in [8], [9].

$M^\lambda$ the permutation module (on $\lambda$-tabloids);

$S^\lambda$ the Specht module;

$D^\lambda$ the irreducible module (if $\lambda$ is $p$-regular);

$Y^\lambda$ the Young module.

**Definition 1.1.** Let $\{t_1, t_2, \dots, t_u\}$ be a finite sequence with $t_i = +1$ or $t_i = -1$. We call such a sequence a *sign* sequence. (We consider the empty sequence as a sign sequence — this corresponds to the case $u = 0$).

**Definition 1.2.** Let $T = \{t_1, t_2, \dots, t_u\}$ be a sign sequence.

(i) We call this sequence *latticed* if and only if $\sum_{i=1}^{j} t_i \geq 0$ for all $j = 1, 2, \dots, u$. The empty sequence is latticed by definition.

(ii) We define $\nu(T)$ by setting

$$\nu(T) = 1 + \Big|\{i \mid 1 \leq i \leq u,\ t_i = +1, \text{ and } \{t_{i+1}, t_{i+2}, \dots, t_u\} \text{ is latticed}\}\Big|,$$

if $T$ is latticed and non-empty;
$\nu(T) = 1$, if $T$ is empty; and
$\nu(T) = 0$, if $T$ is not latticed.

*Example.* Let $T = \{+1, -1, +1\}$. Then $\nu(T) = 2$. Indeed, the empty sequence is latticed so

$$\{i \mid 1 \leq i \leq 3,\ t_i = +1,\ \{t_{i+1}, t_{i+2}, \dots, t_u\} \text{ is latticed}\} = \{3\}.$$

Let $T = \{-1, +1, +1, +1, +1\}$. Then $\nu(T) = 0$ since $T$ is not latticed.

**Definition 1.3.** Let $\lambda$ be a partition of $n$ written in the form (1). Assume $0 \leq i < j \leq m$. Let $\{k_1 > k_2 > \cdots > k_u\}$ be the set of all integers $k$ such that $i < k < j$ and one of the following conditions holds:

(i) $l_k > l_{k+1}$ and $l_k - k \equiv l_j - j$;
(ii) $l_k < l_{k-1}$ and $l_k - k + 1 \equiv l_j - j$.

Now put $T_{i,j}(\lambda) = \{t_1, t_2, \dots, t_u\}$ where $t_v = +1$, if the condition (i) holds for $k = k_v$, and $t_v = -1$, if the condition (ii) holds for $k = k_v$, $v = 1, 2, \dots, u$.

Also put $T_j(\lambda) = T_{0,j}(\lambda)$.

*Remark.* (a) Clearly at most one of the conditions (i) and (ii) in Definition 1.3 holds for a fixed $k = k_v$. So the sequence $T_{i,j}(\lambda)$ is well-defined.

(b) $T_{i,j}(\lambda)$ may be empty.

**Theorem 1.4.** *Let $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ be a $p$-regular partition of $n$. Assume $1 \leq j \leq m$, $l_j > l_{j+1}$ ($l_{m+1}$ is interpreted as $0$) and $\lambda(j)$ is $p$-regular. Then*

$$\left[D^{\lambda} \downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}\right] = \nu(T_j(\lambda)).$$

*Remark.* The sequence $T_j(\lambda)$ is latticed if and only if a removable node $A = (j, l_j)$ is normal in the sense of [14] and [15]. Recall that the residue, $\operatorname{res} C$, of a node $C = (r, s)$ is defined as $(s - r) \pmod p$. Now Theorem 1.4 can be reformulated as follows. The multiplicity $[D^{\lambda} \downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}]$ is equal to 1+(the number of the normal nodes $B$ above $A$ such that $\operatorname{res} B = \operatorname{res} A$), if $A$ is normal, and is equal to 0 otherwise.

*Example.* (a) Let $p = 3$, $\lambda = (5, 3, 2)$. The corresponding Young diagram then looks as follows (we have indicated the residues of all removable nodes and of some indent nodes, cf. Remark after Theorem 1.4 and [15]).

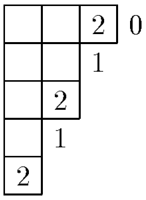
| | | | | | |
|---|---|---|---|---|---|
| | | | | 1 | 2 |
| | | 1 | 2 | | |
| | 2 | | | | |

The sequence $T_3(\lambda)$ is $\{-1, -1\}$. It is not latticed so $D^{(5,3,1)}$ does not occur in $D^{(5,3,2)} \downarrow_{\Sigma_9}$, in view of Theorem 1.4. The sequence $T_2(\lambda) = \{+1\}$. We have $\nu(T_2(\lambda)) = 2$. So $[D^{(5,3,2)} \downarrow_{\Sigma_9} : D^{(5,2,2)}] = 2$. The sequence $T_1(\lambda)$ is empty. So $[D^{(5,3,2)} \downarrow_{\Sigma_9} : D^{(4,3,2)}] = 1$. Moreover, by [15] or [5], $D^{(5,3,2)} \otimes \mathrm{sgn}_{10} \cong D^{(6,2,2)}$. Applying Theorem 1.4 as above, we have $[D^{(6,2,2)} \downarrow_{\Sigma_9} : D^{(5,2,2)}] = 1$. However $D^{(5,2,2)} \otimes \mathrm{sgn}_9 \cong D^{(6,2,1)}$. Therefore $[D^{(5,3,2)} \downarrow_{\Sigma_9} : D^{(6,2,1)}] = 1$. Summarizing, we obtain the following composition factors and multiplicities for $D^{(5,3,2)} \downarrow_{\Sigma_9}$.

| Factor | $D^{(4,3,2)}$ | $D^{(5,2,2)}$ | $D^{(6,2,1)}$ |
|---|---|---|---|
| Multiplicity | 1 | 2 | 1 |

It follows from [25] that this is complete information about composition factors of the restriction $D^{(5,3,2)} \downarrow_{\Sigma_9}$. We add that by [15, 3.1], this restriction is indecomposable, self-dual and has a simple socle and a simple head both isomorphic to $D^{(5,2,2)}$.

(b) Let $p = 3$, $\lambda = (3, 2, 2, 1, 1)$. The Young diagram is




This is an example where Theorem 1.4 is less effective. A reason for this is that $\lambda(j)$ is 3-regular only for $j = 5$. If $j = 5$ we have $T_j(\lambda) = \{+1, +1\}$. So $\nu(T_5(\lambda)) = 3$. Thus, $[D^{(3,2,2,1,1)}\downarrow_{\Sigma_8}: D^{(3,2,2,1)}] = 3$. Now apply the "sign trick". We have $D^{(3,2,2,1,1)} \otimes \mathrm{sgn}_9 \cong D^{(5,3,1)}$. An application of Theorem 1.4 gives the equalities $[D^{(5,3,1)}\downarrow_{\Sigma_8}: D^{(4,3,1)}] = 1$ and $[D^{(5,3,1)}\downarrow_{\Sigma_8}: D^{(5,2,1)}] = 2$. Since $D^{(4,3,1)} \otimes \mathrm{sgn}_8 \cong D^{(7,1)}$ and $D^{(5,2,1)} \otimes \mathrm{sgn}_8 \cong D^{(4,2,2)}$, we conclude that $[D^{(3,2,2,1,1)}\downarrow_{\Sigma_8}: D^{(7,1)}] = 1$ and $[D^{(3,2,2,1,1)}\downarrow_{\Sigma_8}: D^{(4,2,2)}] = 2$. Summarizing, we have:

| Factor | $D^{(3,2,2,1)}$ | $D^{(7,1)}$ | $D^{(4,2,2)}$ |
|---|---|---|---|
| Multiplicity | 3 | 1 | 2 |

As the computer computations in [25] show, the complete set of the composition factors looks as follows.

| Factor | $D^{(3,2,2,1)}$ | $D^{(7,1)}$ | $D^{(4,2,2)}$ | $D^{(4,4)}$ |
|---|---|---|---|---|
| Multiplicity | 3 | 1 | 2 | 1 |

Thus we "lose" $D^{(4,4)}$ (which is actually the sign representation)!

By [15, 3.1], the restriction $D^{(3,2,2,1,1)}\downarrow_{\Sigma_8}$ is indecomposable, self-dual and has a simple socle and a simple head both isomorphic to $D^{(3,2,2,1)}$.

*Remark.* (a) The following result is proved in [7, Theorem C] and [12, 3.3]. Let

$$\lambda = (\lambda_1^{a_1}, (\lambda_1 - 1)^{p-1}, \dots, (\lambda_1 - r + 1)^{p-1}, \lambda_{r+1}^{a_{r+1}}, \dots, \lambda_d^{a_d})$$

be a $p$-regular partition of $n$, with $a_{r+1} \neq p - 1$ or $\lambda_{r+1} < \lambda_1 - r$. Put $j = a_1 + (r-1)(p-1)$. Then $[D^\lambda \downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}] = r$.

This is a special case of Theorem 1.4 since

$$T_j(\lambda) = \{\underbrace{+1, \dots, +1}_{r-1 \text{ times}}\},$$

hence $\nu(T_j(\lambda)) = r$ for our $j$.

It would be important to describe the class of the partitions $\lambda$ for which our method provides a complete set of composition factors of the restriction $D^\lambda\downarrow_{\Sigma_{n-1}}$ and all the composition factors of $D^\lambda\downarrow_{\Sigma_{n-1}}$ again belong to this class. For such partitions the lower bound given in Corollary 1.6 below provides the exact value of the dimension of $D^\lambda$. In [20] (cf. also [16]), a class of partitions is described, for which a much weaker lower bound obtained in [15, 2.2] gives the exact value for the dimension. Already that class is not too slim. This allows one to hope that the lower bound for dim $D^\lambda$ given in Corollary 1.6 is actually *equal* to dim $D^\lambda$ for some reasonably wide class of modules. But not in general, of course.

Recall that we denote by $b_n$ the Mullineux bijection on the set of the $p$-regular partitions of $n$.

**Definition 1.5.** We define the *multiple $p$-Young graph $YG$* as the oriented graph with vertices labelled by all $p$-regular partitions (of all positive integers). Two vertices $\lambda$ and $\mu$ are connected by $i$ arrows going from $\mu$ to $\lambda$ if and only if one of the following conditions holds.

(i) $\mu = \lambda(j)$ and $\nu(T_j(\lambda)) = i$ for some $j$;

(ii) $\mu = b_{n-1}\Big((b_n(\lambda))(k)\Big)$ and $\nu\Big(T_k(b_n(\lambda))\Big) = i$ for some $k$ such that $\big(b_n(\lambda)\big)(k)$ is $p$-regular.

The following result readily follows from Theorem 1.4 by induction on $n$, cf. the proof of [15, 2.2].

**Corollary 1.6.** $\dim D^\lambda \geq \sharp\{$*paths from* $(1)$ *to* $\lambda$ *in* $YG\}$.

We now define the notions necessary to describe some decomposition numbers of $\Sigma_n$.

**Definition 1.7.** Let $J \neq \emptyset$ be a finite set of integers. Then $J$ can be represented in the form

$$J = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_N, c_N]$$

where $b_{i+1} - c_i > 1$ for all $i = 1, 2, \ldots, N-1$. We call this representation *the decomposition into the union of connected components.*

**Definition 1.8.** Let $T = \{t_1, t_2, \ldots, t_u\}$ be a sign sequence.

(i) For $1 \leq i \leq j \leq u$ we denote by $T(i,j)$ the subsequence $\{t_i, t_{i+1}, \ldots, t_j\}$.

(ii) Let $J$ be a subset in $[1, u]$. We call $J$ a *latticed subset* (in $[1, u]$ for $T$) if and only if the following two conditions hold.

  (a) $J \subseteq \{i \mid t_i = -1\}$;

  (b) Let $[1, u] \setminus J = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_N, c_N]$ be the decomposition into the union of connected components. Then the sequences $T(b_i, c_i)$, $i = 1, 2, \ldots, N$, are latticed.

(iii) We denote by $q(T)$ the number of the latticed subsets in $[1, u]$ for $T$.

*Remark.* (a) For $T = \emptyset$ we have $q(T) = 1$ (the empty subset is interpreted as the only latticed subset in $[1, 0] = \emptyset$).

(b) In Lemma 6.7 we provide a recurrence formula for $q(T)$ (in a more general setting of quasi-latticed subsets).

**Definition 1.9.** Let $1 \leq i < j \leq m$. We define an integer $Q_{i,j}(\lambda)$ as follows.

We put $Q_{i,j}(\lambda) = 0$ if $l_j - j \not\equiv l_i - i + 1$.

If $l_j - j \equiv l_i - i + 1$ we put $Q_{i,j}(\lambda) = q(T_{i,j}(\lambda))$ (cf. Definitions 1.3, 1.8(iii)).

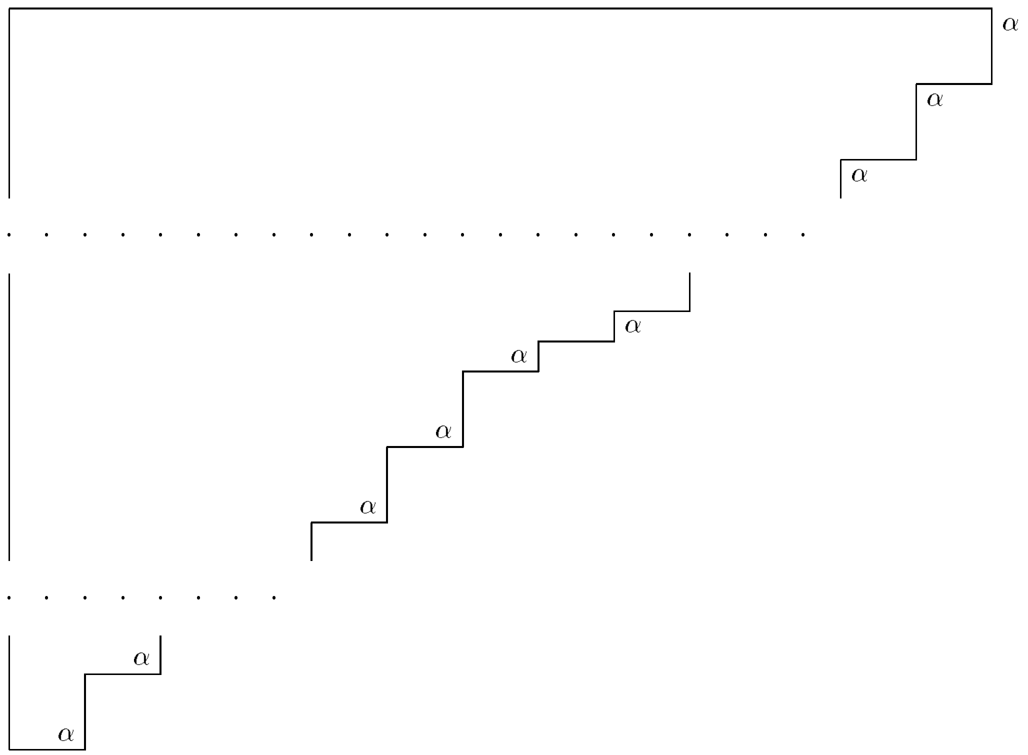
Recall the partition $\lambda(i,j)$ defined in (3).

**Theorem 1.10.** *Let* $\lambda = (l_1 \geq l_2 \geq \dots l_m > 0)$ *be a partition of* $n$, $1 \leq i < j \leq m$, $l_j > l_{j+1}$ *and* $l_i < l_{i-1}$ *(*$l_0$ *and* $l_{m+1}$ *are interpreted as* $+\infty$ *and* $0$*, respectively). Assume the partition* $\lambda(i,j)$ *is* $p$*-regular. Then*

$$\left[S^\lambda : D^{\lambda(i,j)}\right] = Q_{i,j}(\lambda).$$

*Remark.* (a) Let $A = (j, l_j)$, $B = (i, l_i + 1)$. Then the Young diagram of $\lambda(i,j)$ is obtained from that of $\lambda$ by removing $A$ and adding $B$. Now the condition $l_j - j \equiv l_i - i + 1$ in Definition 1.9 means $\operatorname{res} A = \operatorname{res} B$. By the Nakayama Conjecture (cf. [10, 6.1.21, 2.7.41]) this is a necessary condition for $[S^\lambda : D^{\lambda(i,j)}]$ to be non-zero. (This condition is also sufficient in view of [8, 24.10].)

(b) The decomposition numbers $[S^\lambda : D^{\lambda(i,j)}]$ can take *any* non-negative values. Indeed there exists many $p$-regular partitions $\lambda$ such that the top $d+1$ indent nodes and the bottom $d+1$ removable nodes have the same residue, say $\alpha$, while all the other removable and indent nodes have residues different from $\alpha$, for example as in the following picture.




Then the sequence $T_{1,m}(\lambda) = \{\underbrace{+1, \dots, +1}_{d \text{ times}}, \underbrace{-1, \dots, -1}_{d \text{ times}}\}$. Therefore $Q_{1,m}(\lambda) = d + 1$.

It is explained in Section 11 how to identify the multiplicities $[M^\lambda : Y^{\lambda(i,j)}]$ with weight multiplicities of the form $L(\mu)_{\lambda-\alpha}$ where $\alpha$ is a root. The latter multiplicities are described below in Theorem 1.14.

We introduce some standard notation. Details can be found in [24], [11] and [2]. From now on till the end of this section we assume that $K$ is *algebraically closed.* We denote by

$\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ a base in the root system $A_l$;
$\{\omega_1, \omega_2, \dots, \omega_l\}$ the corresponding fundamental weights;
$A_l(K) \cong \mathrm{SL}_{l+1}(K)$ the universal Chevalley group of type $A_l$ over $K$;
$V(\lambda)$ the Weyl module over $A_l(K)$ with highest weight $\lambda$;
$L(\lambda)$ the irreducible module over $A_l(K)$ with highest weight $\lambda$;
$L_\mu$ the $\mu$-weight space of a rational $KA_l(K)$-module $L$.

We fix a dominant weight

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_l\omega_l \qquad (\lambda_1, \lambda_2, \dots, \lambda_l \in \mathbb{Z}_+).$$

Note that any positive root in $A_l$ has a form

$$\alpha(r, s) = \alpha_r + \alpha_{r+1} + \cdots + \alpha_s$$

with $1 \leq r \leq s \leq l$.

We define the constants $\mathcal{B}(i, j)$, $\mathcal{C}(i, j)$ which play a crucial role in this paper (we also used them in [14]).

**Definition 1.11.** For $1 \leq i \leq j \leq l$ we set

$$\mathcal{B}(i, j) = \mathcal{B}_\lambda(i, j) = j - i + \sum_{k=i}^{j} \lambda_k$$

$$\mathcal{C}(i, j) = \mathcal{C}_\lambda(i, j) = j - i + \sum_{k=i}^{j-1} \lambda_k.$$

From now on we assume that $r, s$ are integers with $1 \leq r \leq s \leq l$.

We first give an enumerative description of $\dim L(\lambda)_{\lambda-\alpha}$.

**Definition 1.12.** Let $M, N \subseteq \mathbb{Z}$. We call an injective map $d : M \to N$ a *weakly increasing* (respectively, *weakly decreasing*) *injection* if $m \leq d(m)$ (respectively, $m \geq d(m)$) for all $m \in M$.

In the following definition we introduce a class of subsets of $(r, s]$ which are used to label a basis of the weight space $L(\lambda)_{\lambda-\alpha(r,s)}$, see Theorem 8.6.

**Definition 1.13.** Let $J \subseteq (r, s]$. We call $J$ *non-essential* (in $(r, s]$ for $\lambda$) if and only if there exist $t, u \in [r, s+1] \setminus J$, with $t < u$, and a weakly increasing injection $d : \{t\} \cup ((t, u) \cap J) \to [t, u)$, such that $\mathcal{B}(k, d(k)) \equiv 0$ for all $k \in \{t\} \cup ((t, u) \cap J)$.

Otherwise $J$ is called *essential* (in $(r, s]$ for $\lambda$).

We denote by $\mathbf{E}(r, s) = \mathbf{E}_\lambda(r, s)$ the set of all essential subsets in $(r, s]$, and put $\varepsilon(r, s) = |\mathbf{E}(r, s)|$.

*Example.* (a) The empty set is non-essential in $(r, s]$ if and only if $\mathcal{B}(t, d) \equiv 0$ for some $t, d$ with $r \leq t \leq d \leq s$.

(b) Let $j \in (r, s]$. The set $J = \{j\}$ is non-essential in $(r, s]$ if and only if at least one of the following three conditions holds.

(i) $\mathcal{B}(t,d) \equiv 0$ for some $t,d$ with $j < t \le d \le s$;
(ii) $\mathcal{B}(t,d) \equiv 0$ for some $t,d$ with $r \le t \le d \le j-2$;
(iii) there exist $t \in [r, j-1]$ and two distinct integers $d(t), d(j)$ with $d(t) \in [t,s]$, $d(j) \in [j,s]$ and $\mathcal{B}(t,d(t)) \equiv 0$, $\mathcal{B}(j,d(j)) \equiv 0$.

**Theorem 1.14.** *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \dots + \lambda_l\omega_l$ be a dominant weight for the root system $A_l$, $1 \le r \le s \le l$, $\alpha = \alpha_r + \alpha_{r+1} + \dots + \alpha_s$ be a (positive) root, $\mu = \lambda - \alpha$. Then*

$$\dim L(\lambda)_\mu = \varepsilon(r,s).$$

*Remark.* (a) In fact Theorem 1.14 is a special case of Theorems 8.6 and 8.8. In these theorems we construct two distinct bases of the weight space $L(\lambda)_\mu$ with elements labelled by the essential subsets. Another basis, whose elements are labelled by the so-called $\sigma$-subsets, is constructed in Theorem 8.9.

(b) Below we provide explicit formulas for $\dim L(\lambda)_{\lambda-\alpha}$ (see Theorems 1.18 and 1.19).

**Definition 1.15.** Let $M \subseteq (r,s]$. We say that a set $Z \subseteq (r,s]$ *compensates* $M$ (in $(r,s]$ for $\lambda$) if and only if there exists a weakly decreasing injection $z : M \to Z$ such that $\mathcal{B}(z(m),m) \equiv 0$ for all $m \in M$.

We say that $M$ *is compensated* (in $(r,s]$ for $\lambda$) if and only if there exists a subset $Z \subseteq (r,s]$ which compensates $M$ (in $(r,s]$ for $\lambda$).

**Definition 1.16.** Let $r \le i \le s$. We call $i$ *normal in* $[r,s]$ (for $\lambda$) if and only if $\lambda_i \not\equiv 0$ and the set $M = \{m \in (i,s] \mid \mathcal{B}(i,m) \equiv 0\}$ is compensated in $(r,s]$ (for $\lambda$).

We set

$$\sigma(i,s) = \sigma_\lambda(i,s) = \begin{cases} 1 & \text{if } i \text{ is normal in } [r,s], \\ 0 & \text{otherwise.} \end{cases}$$

We also set

$$\sigma(i,j) = \begin{cases} 1 & \text{if } j = i-1, \\ -1 & \text{if } j = i-2, \\ 0 & \text{if } j < i-2. \end{cases}$$

*Remark.* (a) Clearly, $i$ is normal in $[r,s]$ if and only if it is normal in $[i,s]$.

(b) Assume that $\lambda_r \not\equiv 0$, and let

$$\{r = i_1 < i_2 < \dots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}.$$

Then $r$ is normal in $[r,s]$ if and only if for all $j \in (1,k]$ with $\mathcal{B}(r,i_j) \equiv 0$ there exist $d(j)$, all distinct, such that $1 < d(j) < j$ and $\mathcal{B}(i_{d(j)}, i_j) \equiv 0$. This observation made in Lemma 3.3 is very convenient when those $i$ with $\lambda_i \not\equiv 0$ are known.

**Definition 1.17.** (i) We define an $(s-r+1)$ by $(s-r+1)$ matrix $\mathcal{M}(r,s) = \mathcal{M}_\lambda(r,s)$ as follows.

$$\mathcal{M}(r,s) = \begin{bmatrix} \sigma(r,r-1) & \sigma(r,r) & \cdots & \sigma(r,s-2) & \sigma(r,s) \\ \sigma(r+1,r-1) & \sigma(r+1,r) & \cdots & \sigma(r+1,s-2) & \sigma(r+1,s) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \sigma(s,r-1) & \sigma(s,r) & \cdots & \sigma(s,s-2) & \sigma(s,s) \end{bmatrix}.$$

(ii) Let

$$\{i_1 < i_2 < \cdots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}.$$

We define a matrix $\mathcal{A}(r,s) = \mathcal{A}_\lambda(r,s)$ as a $k$ by $k$ matrix with entries $a_{t,u}$ given by

$$a_{t,u} = \begin{cases} \sigma(i_t, i_{u-1}) + (i_{u+1} - i_u - 1)\sigma(i_t, i_u), & \text{if } 1 \le t < u < k; \\ \sigma(i_t, i_k), & \text{if } u = k; \\ -1, & \text{if } u = t-1; \\ 0, & \text{otherwise.} \end{cases}$$

*Example.* (i)

$$\mathcal{M}(r,r) = [\sigma(r,r)],$$

$$\mathcal{M}(r,r+1) = \begin{bmatrix} \sigma(r,r-1) & \sigma(r,r+1) \\ \sigma(r+1,r-1) & \sigma(r+1,r+1) \end{bmatrix} = \begin{bmatrix} 1 & \sigma(r,r+1) \\ -1 & \sigma(r+1,r+1) \end{bmatrix},$$

$$\begin{aligned} \mathcal{M}(r,r+2) &= \begin{bmatrix} \sigma(r,r-1) & \sigma(r,r) & \sigma(r,r+2) \\ \sigma(r+1,r-1) & \sigma(r+1,r) & \sigma(r+1,r+2) \\ \sigma(r+2,r-1) & \sigma(r+2,r) & \sigma(r+2,r+2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & \sigma(r,r) & \sigma(r,r+2) \\ -1 & 1 & \sigma(r+1,r+2) \\ 0 & -1 & \sigma(r+2,r+2) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \mathcal{M}(r,r+3) &= \begin{bmatrix} \sigma(r,r-1) & \sigma(r,r) & \sigma(r,r+1) & \sigma(r,r+3) \\ \sigma(r+1,r-1) & \sigma(r+1,r) & \sigma(r+1,r+1) & \sigma(r+1,r+3) \\ \sigma(r+2,r-1) & \sigma(r+2,r) & \sigma(r+2,r+1) & \sigma(r+2,r+3) \\ \sigma(r+3,r-1) & \sigma(r+3,r) & \sigma(r+3,r+1) & \sigma(r+3,r+3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & \sigma(r,r) & \sigma(r,r+1) & \sigma(r,r+3) \\ -1 & 1 & \sigma(r+1,r+1) & \sigma(r+1,r+3) \\ 0 & -1 & 1 & \sigma(r+2,r+3) \\ 0 & 0 & -1 & \sigma(r+3,r+3) \end{bmatrix}. \end{aligned}$$

(ii) Denote $\sigma(i_t, i_u)$ by $\sigma_{t,u}$, and put $x_t = i_t - i_{t-1}$. Then for $k = 1$ we have $\mathcal{A}(r,s) = [\sigma_{1,1}]$; for $k = 2$ we have

$$\mathcal{A}(r,s) = \begin{bmatrix} x_2 & \sigma_{1,2} \\ -1 & \sigma_{2,2} \end{bmatrix};$$

for $k = 3$ we have

$$\mathcal{A}(r,s) = \begin{bmatrix} x_2 & \sigma_{1,1} + (x_3 - 1)\sigma_{1,2} & \sigma_{1,3} \\ -1 & x_3 & \sigma_{2,3} \\ 0 & -1 & \sigma_{3,3} \end{bmatrix};$$

for $k = 4$ we have

$$\mathcal{A}(r,s) = \begin{bmatrix} x_2 & \sigma_{1,1} + (x_3 - 1)\sigma_{1,2} & \sigma_{1,2} + (x_4 - 1)\sigma_{1,3} & \sigma_{1,4} \\ -1 & x_3 & \sigma_{2,2} + (x_4 - 1)\sigma_{2,3} & \sigma_{2,4} \\ 0 & -1 & x_4 & \sigma_{3,4} \\ 0 & 0 & -1 & \sigma_{4,4} \end{bmatrix}.$$

**Theorem 1.18.** *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \dots + \lambda_l\omega_l$ be a dominant weight for the root system $A_l$ and $\alpha = \alpha_r + \alpha_{r+1} + \dots + \alpha_s$ be a positive root. Then*

$$\dim L(\lambda)_{\lambda-\alpha} = \det \mathcal{M}_\lambda(r,s) = \det \mathcal{A}_\lambda(r,s).$$

In fact, we first prove recurrence formulas for $\dim L(\lambda)_{\lambda-\alpha}$ given in in Theorem 1.19 below, and then observe that they are just cofactor expansions of $\det \mathcal{M}_\lambda(r,s)$ and $\det \mathcal{A}_\lambda(r,s)$ along the first rows! Part (ii) of Theorem 1.19 is convenient when those $i$ with $\lambda_i \not\equiv 0$ are known. Other recurrence formulas for $\dim L(\lambda)_{\lambda-\alpha}$ can be found in Corollary 8.10. Below we also give another expression of $\varepsilon(r,s) = \dim L(\lambda)_{\lambda-\alpha}$ as a determinant, see (59).

**Theorem 1.19.** *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \dots + \lambda_l\omega_l$ be a dominant weight for the root system $A_l$, $1 \le r \le s \le l$. For a positive root $\alpha(i,j) = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$ we denote $d(i,j) = \dim L(\lambda)_{\lambda-\alpha(i,j)}$. Then*

(i) $d(r,s) = \sigma(r,s) + \sum_{j=r+1}^{s} \sigma(r, j-2) d(j,s)$.

(ii) *Let $\{i_1 < i_2 < \dots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}$. If $I = \emptyset$ then $d(r,s) = 0$. If $I \neq \emptyset$ then*

$$\begin{aligned} d(r,s) = &\sigma(i_1, i_k) + (i_2 - i_1) d(i_2, i_k) \\ &+ \sum_{j=3}^{k} \Big( (i_j - i_{j-1} - 1)\sigma(i_1, i_{j-1}) + \sigma(i_1, i_{j-2}) \Big) d(i_j, i_k). \end{aligned}$$

*Remark.* Let $\{i_1 < i_2 < \dots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}$.

If $k \le 3$ then $\dim L(\lambda)_{\lambda-\alpha(r,s)}$ was known previously.

For $k = 0, 1$ the problem is trivial.

For $k = 2$ the answer can be found in [21, 8.6] (see also [23, p. 25]).

For $k = 3$ the weight multiplicity has also been described by Seitz (unpublished).

We provide the explicit answers for $k \le 4$ in the next corollary (the proof of the corollary is a straightforward application of Theorem 1.18 (or Theorem 1.19(ii)) and Remark (b) after Definition 1.16).

**Corollary 1.20.** *Let*

$$\{i_1 < i_2 < \dots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}.$$

*Denote $d(r,s) = \dim L(\lambda)_{\lambda-\alpha(r,s)}$, $\mathcal{B}_{t,u} = \mathcal{B}(i_t, i_u)$, $x_t = i_t - i_{t-1}$. Then*

(i) *If* $k = 0$ *then* $d(r, s) = 0$.
(ii) *If* $k = 1$ *then* $d(r, s) = 1$.
(iii) [21, 8.6] *Let* $k = 2$. *Then*

$$d(r,s) = \begin{cases} x_2, & \text{if } \mathcal{B}_{1,2} \equiv 0, \\ x_2 + 1, & \text{otherwise.} \end{cases}$$

(iv) *Let* $k = 3$. *Then*

$$d(r,s) = \begin{cases} x_2 x_3 + 1, & \text{if } \mathcal{B}_{1,2} \equiv 0,\ \mathcal{B}_{2,3} \equiv 0; \\ x_2(x_3 + 1) + 1, & \text{if } \mathcal{B}_{1,2} \equiv 0,\ \mathcal{B}_{2,3} \not\equiv 0; \\ (x_2 + 1)x_3 + 1, & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\ \mathcal{B}_{2,3} \equiv 0; \\ (x_2 + 1)(x_3 + 1) - 1, & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\ \mathcal{B}_{2,3} \not\equiv 0,\ \mathcal{B}_{1,3} \equiv 0; \\ (x_2 + 1)(x_3 + 1), & \text{if } \mathcal{B}_{1,2} \not\equiv 0,\ \mathcal{B}_{2,3} \not\equiv 0,\ \mathcal{B}_{1,3} \not\equiv 0. \end{cases}$$

(v) *Let* $k = 4$. *Then the value of* $d(r, s)$ *can be found from the following table.*

| $d(r,s)$ | $\mathcal{B}_{1,2}$ | $\mathcal{B}_{1,3}$ | $\mathcal{B}_{1,4}$ | $\mathcal{B}_{2,3}$ | $\mathcal{B}_{3,4}$ | $\mathcal{B}_{2,4}$ |
|---|---|---|---|---|---|---|
| $x_2x_3x_4 + x_2 + x_4$ | $\equiv 0$ | any | any | $\equiv 0$ | $\equiv 0$ | any |
| $(x_2+1)x_3x_4 + x_2 + x_4 + 1$ | $\not\equiv 0$ | any | any | $\equiv 0$ | $\equiv 0$ | any |
| $x_2x_3(x_4+1) + x_2 + x_4 + 1$ | $\equiv 0$ | any | any | $\equiv 0$ | $\not\equiv 0$ | any |
| $(x_2+1)x_3(x_4+1) + x_2 + x_4 + 1$ | $\not\equiv 0$ | any | $\not\equiv 0$ | $\equiv 0$ | $\not\equiv 0$ | any |
| as above | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ | $\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ |
| $(x_2+1)x_3(x_4+1) + x_2 + x_4$ | $\not\equiv 0$ | $\equiv 0$ | $\equiv 0$ | $\equiv 0$ | $\not\equiv 0$ | any |
| as above | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ | $\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ |
| $x_2(x_3+1)x_4 + x_2 + x_4$ | $\equiv 0$ | any | any | $\not\equiv 0$ | $\equiv 0$ | any |
| $x_2(x_3+1)x_4 + x_2 + x_3x_4 + 1$ | $\not\equiv 0$ | $\equiv 0$ | any | $\not\equiv 0$ | $\equiv 0$ | any |
| $(x_2+1)(x_3+1)x_4 + x_2 + 1$ | $\not\equiv 0$ | $\not\equiv 0$ | any | $\not\equiv 0$ | $\equiv 0$ | any |
| $x_2x_3(x_4+1) + (x_2+1)x_4 + 1$ | $\equiv 0$ | any | any | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ |
| $x_2(x_3+1)x_4 + x_2x_3 + x_3(x_4+1) + 1$ | $\not\equiv 0$ | $\equiv 0$ | any | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ |
| $(x_2+1)x_3(x_4+1) + (x_2+1)x_4 + 1$ | $\not\equiv 0$ | $\not\equiv 0$ | any | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ |
| $x_2(x_3+1)(x_4+1) + (x_4+1)$ | $\equiv 0$ | any | any | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ |
| $x_2(x_3+1)(x_4+1) + x_3(x_4+1) + 1$ | $\not\equiv 0$ | $\equiv 0$ | any | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ |
| $x_2(x_3+1)(x_4+1) + x_3(x_4+1) + x_4$ | $\not\equiv 0$ | $\not\equiv 0$ | $\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ |
| $(x_2+1)(x_3+1)(x_4+1)$ | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ | $\not\equiv 0$ |

Now assume that those $i \in [r, s]$ with $\lambda_i \not\equiv 0$ are fixed. What is the *minimal* value for $\dim L(\lambda)_{\lambda - \alpha(r,s)}$? Clearly, the *maximal* value is equal to the corresponding dimension in the case of zero characteristic, i.e. $\dim V(\lambda)_{\lambda-\alpha(r,s)}$. And it is well known that if

$$\emptyset \neq \{i \in [r, s] \mid \lambda_i \not\equiv 0\} = \{i_1 < i_2 < \cdots < i_k\}$$

then

$$\dim V(\lambda)_{\lambda - \alpha(r,s)} = \prod_{j=1}^{k-1} (i_j - i_{j-1} + 1),$$

see for example [12, 1.6].

**Definition 1.21.** Let $i_1 < i_2 < \cdots < i_k$ be integers. We define the integers $f(i_j, i_{j+1}, \dots, i_k)$, $j = k, k-1, \dots, 1$, by setting

$$f(i_k) = 1;$$
$$f(i_{k-1}, i_k) = i_k - i_{k-1};$$
$$f(i_j, i_{j+1}, \dots, i_k) = (i_{j+1} - i_j) f(i_{j+1}, i_{j+2}, \dots, i_k) + f(i_{j+2}, i_{j+3}, \dots, i_k),$$
$$j = k-2, k-3, \dots, 1.$$

*Example.* Let $i_1 = r,\ i_2 = r+1,\ i_3 = r+2, \dots, i_{s-r+1} = s$. Then $f(r, r+1, \dots, s) = f_{s-r+1}$ where $f_j$ denotes the $j$-th Fibonacci number defined by $f_1 = 1,\ f_2 = 1$ and $f_j = f_{j-1} + f_{j-2}$ for $j \geq 3$.

**Proposition 1.22.** *Assume that*

$$\{i_1 < i_2 < \cdots < i_k\} = \{i \in [r, s] \mid \lambda_i \not\equiv 0\} \neq \emptyset.$$

*Then*

$$\dim L(\lambda)_{\lambda - \alpha(r,s)} \geq f(i_1, i_2, \dots, i_k).$$

*Moreover,*

$$\dim L(\lambda)_{\lambda - \alpha(r,s)} = f(i_1, i_2, \dots, i_k).$$

*if and only if*

$$\mathcal{B}(i_j, i_{j+1}) \equiv 0 \textit{ for all } j = 1, 2, \dots, k-1. \tag{4}$$

The description of the multiplicities $[V(\lambda) : L(\lambda - \alpha)]$ is given in combinatorial terms which are very similar to those used for the description of $\left[S^\lambda : D^{\lambda(i,j)}\right]$, cf. 1.8, 1.9 and 1.10. (In fact, we deduce the latter description from the former one). We have only to consider more general *quasi-latticed* sequences which consist of "+1", "−1" or "0". The exact formulations can be found in Sections 6 and 10. The main result is Theorem 10.6. Corollary 10.5 provides a recurrence formula for $[V(\lambda) : L(\lambda - \alpha)]$.

## Part I. Preliminary combinatorial results

Throughout Part I:

$l$ is a fixed positive integer;

$r$ and $s$ are always supposed to be integers which satisfy $1 \leq r \leq s \leq l$;

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\lambda_i \in \mathbb{Z}_+$, $i = 1, 2, \dots, l$, is an $l$-tuple of non-negative integers (in Part II we shall consider $\lambda$ as a dominant weight for the root system $A_l$).

We define the $l$-tuple $\lambda(i)$ for any $i = 1, 2, \dots, l$ by setting

$$\lambda(i) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_l). \tag{5}$$

(The $l$-tuple $\lambda(i)$ has nothing to do with the partition $\lambda(j)$ from (2)!)

Recall the constants $\mathcal{B}(i,j) = \mathcal{B}_\lambda(i,j)$ and $\mathcal{C}(i,j) = \mathcal{C}_\lambda(i,j)$ defined in 1.11.

## 2. Essential subsets

In 1.13 we have defined the set $\mathbf{E}(r,s) = \mathbf{E}_\lambda(r,s)$ which consists of the essential subsets in $(r,s]$ for $\lambda$. We have also defined the integer $\varepsilon(r,s)$ as the order of $\mathbf{E}(r,s)$.

Recall the $l$-tuples $\lambda(i)$ defined in (5). Put

$$\mathbf{E}'(r,s) = \mathbf{E}_{\lambda(s)}(r,s), \tag{6}$$

$$\varepsilon'(r,s) = \left|\mathbf{E}'(r,s)\right|. \tag{7}$$

The following two lemmas follow immediately from Definition 1.13.

**Lemma 2.1.** *Let $I, J \subseteq (r,s]$ and $I \subseteq J$. If $I$ is essential in $(r,s]$ then $J$ is essential in $(r,s]$.*

**Lemma 2.2.** *Let $J \subseteq (r,s]$ be essential in $(r,s]$ and $q \in (r,s]\setminus J$. Then $J \cap (r,q-1]$ is essential in $(r,q-1]$.*

**Lemma 2.3.** *Let $I = \{i_1 < i_2 < \cdots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}$*

(i) *If $I = \emptyset$ then $\varepsilon(r,s) = 0$;*
(ii) *If $I \neq \emptyset$ then $\varepsilon(r,s) = \varepsilon(i_1, i_k)$.*

*Proof.* (i) $I = \emptyset$ means $\lambda_i \equiv 0$ for $i = r, r+1, \ldots, s$. In this case $(r,s]$ is non-essential in $(r,s]$. Indeed, we can take

$$t = r, \quad u = s+1, \quad d(t) = r, \quad d(j) = j \text{ for all } j \in (r,s]$$

in Definition 1.13, because $\mathcal{B}_\lambda(k,k) = \lambda_k$. Now (i) follows from Lemma 2.1.

(ii) Let $J \subseteq (r,s]$. It is enough to prove that $J$ is essential in $(r,s]$ if and only if $J \supseteq (r,i_1]$, $J \supseteq (i_k,s]$ and $\gamma(J) := J \setminus ((r,i_1] \cup (i_k,s])$ is essential in $(i_1,i_k]$.

Assume $J$ is essential in $(r,s]$. If there exists $q \in (r,i_1]$ with $q \notin J$ then we may take

$$t = i_1, \quad u = q, \quad d(t) = i_1, \quad d(j) = j \text{ for all } j \in J \cap (i_1,q)$$

in Definition 1.13. If there exists $q \in (i_k,s]$ with $q \notin J$ then we may take

$$t = q, \quad u = s+1, \quad d(t) = q, \quad d(j) = j \text{ for all } j \in J \cap (q,s])$$

in Definition 1.13. Thus $J \supseteq (r,i_1]$, $J \supseteq (i_k,s]$.

Assume $\gamma(J)$ is not essential in $(i_1,i_k]$. Then there exist $t$, $u$, $d(t)$ and $d(j)$ as in Definition 1.13. If $t > i_1$ and $u \leq i_k$ this implies that $J$ in non-essential in $(r,s]$, and we are done.

Suppose $t = i_1$, $u \leq i_k$. We choose $t' = r$, $d(j) = j$ for all $j \in [i_1,i_k)$. Then

$$t', \quad u, \quad d(t'), \quad \text{and} \quad d(j) \text{ for } j \in J \cap (t',u)$$

work in Definition 1.13. Therefore $J$ is non-essential in $(r,s]$.

If $t = i_1$, $u = i_k+1$ we need to take $t'$ as above, $u' = s+1$, $d(j) = j$ for $j = i_k+1, i_k+2, \ldots, s$ in Definition 1.13.

The case $t > i_1$, $u = i_k+1$ is considered similarly.

In the other direction, suppose $J \supseteq (r, i_1]$, $J \supseteq (i_k, s]$ and $\gamma(J)$ is essential in $(i_1, i_k]$. If $J$ is non-essential in $(r, s]$ then there exist $t$, $u$, $d(t)$, $d(j)$ as in Definition 1.13. Since $\gamma(J)$ is essential in $(i_1, i_k]$ we have $t = r$ or $u = s+1$ or both, since otherwise $t, u \notin J$ implies $t, u \in (i_1, i_k]$.

Assume $t = r$, $u = s+1$. Since $(i_k, s] \subseteq J \cap (t, u)$ we have $d(j) = j$ for all $j \in (i_k, s]$. So $d(j) \leq i_k$ for all $j \in J \cap (t, i_k + 1)$. Take $u' = i_k + 1$, $t' = i_1$. Then, since $i_1 \in J$ conditions of Definition 1.13 hold for $u'$, $t'$, $d(t')$, $d(j)$ for $j \in J \cap (u', t') = \gamma(J) \cap (u', t')$. So $\gamma(J)$ is non-essential in $(i_1, i_k]$.

The cases $t = r$, $u < s+1$ and $t > r$, $u = s+1$ are considered similarly. □

**Lemma 2.4.** *Let $q \in (r, s]$, $I \in \boldsymbol{E}(r, q-2)$, $J \in \boldsymbol{E}(q, s)$. Then $I \cup \{q\} \cup J \in \boldsymbol{E}(r, s)$. (If $q = r+1$, the statement is interpreted as follows: $J \in \boldsymbol{E}(r+1, s)$ implies $\{r+1\} \cup J \in \boldsymbol{E}(r, s)$).*

*Proof.* Assume the lemma is false. Then by Definition 1.13 there exist
$$t, u \in [r, s+1] \setminus (I \cup \{q\} \cup J),$$
with $t < u$, and a weakly increasing injection
$$d : \{t\} \cup ((t, u) \cap (I \cup \{q\} \cup J)) \to [t, u)$$
such that
$$\mathcal{B}(k, d(k)) \equiv 0$$
for all $k \in \{t\} \cup ((t, u) \cap (I \cup \{q\} \cup J))$.

If $u < q$ it follows from the definition that $I$ is non-essential in $(r, q-2]$, which contradicts our assumption.

If $u > q$ and $t > q$ it follows from the definition that $J$ is non-essential in $(q, s]$, giving a contradiction with the assumption.

Let $u > q$, $t < q$. Then $q \in (t, u) \cap (I \cup \{q\} \cup J)$, so we have all distinct
$$d(q), \quad d(j) \text{ for } j \in (q, u) \cap J,$$
such that
$$q \leq d(q) < u, \quad \mathcal{B}(q, d(q)) \equiv 0$$
and
$$j \leq d(j) < u, \quad \mathcal{B}(j, d(j)) \equiv 0 \text{ for all } j \in (q, u) \cap J.$$
Thus, we may take $r = q$, $t = q$ and $u = u$ in Definition 1.13 to prove that $J$ is non-essential in $(q, s]$, which contradicts the assumption. □

We introduce a lexicographic order "$<$" on the set of all subsets of $(r, s]$.

**Definition 2.5.** Let $I = \{i_a < i_{a-1} < \cdots < i_1\}$, $J = \{j_b < j_{b-1} < \cdots < j_1\}$ be distinct subsets of $(r, s]$. We write $I < J$ if and only if one of the following conditions holds

(i) there exists $c \in [1, \min(a, b)]$ such that $i_c < j_c$ and $i_f = j_f$ for all $f \in [1, c)$;
(ii) $b > a$ and $i_f = j_f$ for all $f \in [1, a]$.

The following lemma follows immediately from Definition 2.5.

**Lemma 2.6.** *Let $I, J \subseteq (r, s]$. Then $I > J$ if and only if $((r, s] \setminus I) < ((r, s] \setminus J)$.*

**Definition 2.7.** Let two subsets $I, J \subseteq (r, s]$ be decomposed into connected components (see Definition 1.7):

$$I = I_1 \cup I_2 \cup \cdots \cup I_H = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_H, c_H],$$
$$J = J_1 \cup J_2 \cup \cdots \cup J_N = [\beta_1, \gamma_1] \cup [\beta_2, \gamma_2] \cup \cdots \cup [\beta_N, \gamma_N].$$

(i) We write $I \ll J$ if and only if for all $v = 1, 2, \ldots, H$ there exist $u_v \in [1, N]$ such that $b_v = \beta_{u_v},\ c_v \leq \gamma_{u_v}$.

(ii) Let $I \ll J$ be as in (i). We define sets $\Delta_v(I, J)$, $v = 0, 1, \ldots, H$ by setting

$$\Delta_0(I, J) = \cup_{w=1}^{u_1 - 1} J_w;$$
$$\Delta_v(I, J) = (J_{u_v} \setminus I_v) \cup \Big( \cup_{w=u_v+1}^{u_{v+1}-1} J_w \Big), \quad v = 1, 2, \ldots, H - 1;$$
$$\Delta_H(I, J) = (J_{u_H} \setminus I_H) \cup \Big( \cup_{w=u_H+1}^{N} J_w \Big).$$

(iii) Let $I \ll J$ be as in (i) and $\Delta_v(I, J)$ be as in (ii). We say that $I$ is linked with $J$ if and only if for all $v \in [0, H]$ there exists a weakly increasing injection

$$d : \Delta_v(I, J) \to (-\infty, b_{v+1} - 2]$$

such that $\mathcal{B}(j, d(j)) \equiv 0$ for all $j \in \Delta_v(I, J)$ ($b_{H+1}$ is interpreted as $s + 2$). We denote $\{d(j) \mid j \in \Delta_v(I, J)\}$ by $\Omega_v(I, J)$.

**Definition 2.8.** For any $J \subseteq (r, s]$ we define the subset $\hat{J} \subseteq (r, s]$ to be the minimal (with respect to "$<$", see 2.5) subset such that $\hat{J} \ll J$ and $\hat{J}$ is linked with $J$.

*Remark.* $\hat{J}$ is well defined since $J$ is always linked with $J$.

**Lemma 2.9.** *Let $I \neq J$ be two essential subsets in $(r, s]$. Then $\hat{I} \neq \hat{J}$.*

*Proof.* Let $\hat{I} = \hat{J}$ and assume without loss of generality that $J > I$. Then $J \not\subseteq I$. So we can choose $t$ to be the maximal integer with $t \in J,\ t \notin I$. Then $J \cap (t, s] \subseteq I \cap (t, s]$. Now $J > I$ implies

$$J \cap (t, s] = I \cap (t, s]. \tag{8}$$

Note that

$$t \notin \hat{J} \tag{9}$$

since otherwise $t \in \hat{J} = \hat{I} \subseteq I$.

Let

$$J = J_1 \cup J_2 \cup \cdots \cup J_N = [\beta_1, \gamma_1] \cup [\beta_2, \gamma_2] \cup \cdots \cup [\beta_N, \gamma_N],$$
$$\hat{J} = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_H, c_H]$$

be decompositions into connected components. By Definition 2.7 we have $b_v = \beta_{u_v}$, $c_v \leq \gamma_{u_v}$ for some $u_v \in [1, N]$, $v = 1, 2, \dots, H$. Let $m \in [1, N]$ be such that $t \in J_m$.

Put $Z_0 = (r, b_1)$, $Z_v = (c_v, b_{v+1})$, $v = 1, 2, \dots, H-1$, $Z_H = (c_H, s]$. Then $(r, s] \setminus \hat{J} = Z_0 \sqcup Z_1 \sqcup \cdots \sqcup Z_H$. By (9), $t \in (r, s] \setminus \hat{J}$. So there exists $f \in [0, H]$ such that $t \in Z_f$.

Set

$$X = [t, \gamma_m] \cup \left( \cup_{w=m+1}^{u_{f+1}-1} J_w \right)$$

where $u_{H+1}$ is interpreted as $N + 1$. We claim that

$$X \subseteq \Delta_f(\hat{J}, J). \tag{10}$$

Indeed, if $f = 0$ then $t < b_1$, $t \in J_m$ imply $m < u_1$ since $J_{u_1} = [\beta_{u_1}, \gamma_{u_1}] = [b_1, \gamma_{u_1}]$. So

$$X = [t, \gamma_m] \cup (\cup_{w=m+1}^{u_1-1} J_w) \subseteq \cup_{w=1}^{u_1-1} J_w = \Delta_0(\hat{J}, J).$$

If $f \in [1, H)$ then $t \in (c_f, b_{f+1})$, $t \in J_m$ imply $u_f \leq m \leq u_{f+1} - 1$ since $\beta_{u_f} = b_f \leq c_f$, $b_{f+1} = \beta_{u_{f+1}}$. So, taking into account that $t > c_f$ we get

$$X = [t, \gamma_m] \cup \left( \cup_{w=m+1}^{u_{f+1}-1} J_w \right) \subseteq (J_{u_f} \setminus [b_f, c_f]) \cup \left( \cup_{w=u_f+1}^{u_{f+1}-1} J_w \right) = \Delta_f(\hat{J}, J).$$

The case $f = H$ is considered similarly.

By Definition 2.7(iii), the embedding (10) implies that for all $j \in X$ there exist $d(j) \in [j, b_{f+1} - 2]$, all distinct, with $\mathcal{B}(j, d(j)) \equiv 0$ ($b_{H+1}$ is interpreted as $s + 2$).

Put $u = b_{f+1} - 1$. Then we have $t \in [r, s] \setminus I$, $u \in (r, s+1] \setminus I$. Besides, using (8), we get

$$(t, u) \cap I = (t, u) \cap J = (t, b_{f+1} - 1) \cap J = (t, \beta_{u_{f+1}} - 1) \cap J = X \setminus \{t\}.$$

Thus we have $t$, $u$, $d(t)$, $d(j)$ for $j \in I \cap (t, u) = X \setminus \{t\}$ as in Definition 1.13. This shows that $I$ is not essential in $(r, s]$ which contradicts the assumptions of the lemma. $\square$

## 3. Normality

We have defined in 1.16 the normal integers $i$ in $[r, s]$ (for $\lambda$). Now we introduce some related notions.

**Definition 3.1.** We define the set $\mathbf{N}_\lambda(r, s)$ as follows. If $r$ is not normal in $[r, s]$ for $\lambda$ we put $\mathbf{N}_\lambda(r, s) = \emptyset$. If $r$ is normal in $[r, s]$ for $\lambda$ set

$$\mathbf{N}_\lambda(r, s) = \{i \in [r, s] \mid i \text{ is normal in } [r, s] \text{ for } \lambda, \text{ and } \mathcal{C}_\lambda(r, i) \equiv 0\}.$$

In what follows we shall denote

$$\mathbf{N}(r, s) = \mathbf{N}_\lambda(r, s),$$
$$\mathbf{N}'(r, s) = \mathbf{N}_{\lambda(s)}(r, s),$$
$$\bar{\mathbf{N}}(r, s) = \mathbf{N}_{\lambda(r)}(r, s).$$

We put

$$\begin{aligned}\nu(r,s) &= \left|\mathbf{N}(r,s)\right|,\\ \nu'(r,s) &= \left|\mathbf{N}'(r,s)\right|,\\ \bar{\nu}(r,s) &= \left|\bar{\mathbf{N}}(r,s)\right|.\end{aligned}$$

*Remark.* If $r$ is normal in $[r,s]$ then $r \in \mathbf{N}(r,s)$ since $\mathcal{C}_\lambda(r,r) = 0$. So $\nu(r,s) \neq 0$ if and only if $r$ is normal in $[r,s]$.

**Lemma 3.2.** *Let $r \leq i \leq t \leq s$. If $i$ is normal in $[r,s]$ then $i$ is normal in $[r,t]$.*

*Proof.* This follows immediately from Definition 1.16. $\square$

**Lemma 3.3.** *Let $\lambda_r \not\equiv 0$, and*

$$\{r = i_1 < i_2 < \cdots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}.$$

*Then $r$ is normal in $[r,s]$ if and only if for all $j \in (1,k]$ with $\mathcal{B}(r,i_j) \equiv 0$ there exist $d(j)$, all distinct, such that $d(j) \in (1,j)$ and $\mathcal{B}(i_{d(j)}, i_j) \equiv 0$.*

*Proof.* This is essentially proved in [14, 4.3], cf. also [14, 4.2]. $\square$

**Lemma 3.4.** *Let $r$ be normal in $[r,s]$, and*

$$\begin{aligned}\{m_t < m_{t-1} < \cdots < m_1\} &= \{m \in (r,s] \mid \mathcal{B}(r,m) \equiv 0\},\\ \{d_u < d_{u-1} < \cdots < d_1\} &= \{d \in (r,s] \mid \mathcal{C}(r,d) \equiv 0\}.\end{aligned}$$

*If $u > t$ then there exists $v \in [1,u]$ such that $d_v$ is normal in $[r,s]$.*

*Proof.* Let $v$ be the least integer in $[1,u]$ such that

$$|\{m_w \mid m_w \geq d_v\}| < v. \tag{11}$$

Then $v$ is well defined since $|\{m_w \mid m_w \geq d_u\}| \leq t < u$. Note that

$$d_v \notin \{m_1, m_2, \ldots, m_t\}. \tag{12}$$

Indeed, let $d_v = m_q$. Then $v > 1$ since otherwise (11) implies $\{m_w \mid m_w \geq d_1\} = \emptyset$ giving a contradiction. If $v > 1$ then by (11) we have $|\{m_q, m_{q-1}, \ldots, m_1\}| < v$, i.e. $q < v$. On the other hand,

$$\{m_w \mid m_w \geq d_{v-1}\} \subseteq \{m_{q-1}, m_{q-2}, \ldots, m_1\}$$

since $d_{v-1} > d_v = m_q$. Hence

$$|\{m_w \mid m_w \geq d_{v-1}\}| \leq q - 1 < v - 1$$

which contradicts the choice of $v$.

From (12) we have $\lambda_{d_v} = \mathcal{B}(r,d_v) - \mathcal{C}(r,d_v) \equiv \mathcal{B}(r,d_v) \not\equiv 0$, i.e.

$$\lambda_{d_v} \not\equiv 0. \tag{13}$$

Now note that

$$|\{m_w \mid m_w > d_v\}| = v - 1. \tag{14}$$

In fact, if $v=1$, this follows from (11). If $v>1$ and $\{m_w \mid m_w > d_v\} < v-1$ then

$$|\{m_w \mid m_w \geq d_{v-1}\}| \leq |\{m_w \mid m_w > d_v\}| < v-1.$$

This contradicts the choice of $v$.

If $i \in [1,u]$ and $m \in (d_i, s]$ then $\mathcal{B}(d_i,m) \equiv \mathcal{C}(r,d_i) + \mathcal{B}(d_i,m) = \mathcal{B}(r,m)$, i.e.

$$\mathcal{B}(d_i,m) \equiv \mathcal{B}(r,m) \quad \text{for any } i \in [1,u] \text{ and } m \in (d_i,s]. \tag{15}$$

Now (15) and (14) imply

$$\{m \in (d_v,s] \mid \mathcal{B}(d_v,m) \equiv 0\} = \{m_{v-1} < m_{v-2} < \cdots < m_1\}. \tag{16}$$

Finally we prove that

$$\{d_1, d_2, \dots, d_{v-1}\} \text{ compensates } \{m_1, m_2, \dots, m_{v-1}\} \text{ in } (d_v, s]. \tag{17}$$

We may assume that $v>1$ since otherwise (17) is trivial. Note that, in view of (15), $d_i < m_j$ implies $\mathcal{B}(d_i,m_j) \equiv 0$ for all $i,j$. So it suffices to prove that $d_i \leq m_i$ for $i=1,2,\dots,v-1$. We prove this by induction on $i$. If $d_1 > m_1$ we get $v=1$ by the definition of $v$. But this contradicts our assumption. Let $i \in [2, v-1]$, and we have proved that $d_j \leq m_j$ for $j = 1,2,\dots,i-1$. If $d_i > m_i$ then $|\{m_w \mid m_w \geq d_i\}| = i-1 < i$, which contradicts the choice of $v$ since $i<v$.

Now $d_v$ is normal in $[r,s]$ by (13), (16) and (17). $\square$

**Lemma 3.5.** *Assume that $r<s$. Then*

$$\nu(r,s) = \begin{cases} \nu(r,s-1)+1 & \text{if } \nu(r,s-1) \neq 0,\ \lambda_s \not\equiv 0,\ \mathcal{C}(r,s) \equiv 0; \\ \nu(r,s-1)-1 & \text{if } \nu(r,s-1) \neq 0,\ \lambda_s \not\equiv 0,\ \mathcal{B}(r,s) \equiv 0; \\ \nu(r,s-1) & \text{otherwise.} \end{cases}$$

*Proof.* Put $\nu = \nu(r,s-1)$. If $\nu=0$ then $r$ is not normal in $[r,s-1]$ (see Remark after 3.1), hence $r$ is not normal in $[r,s]$ by Lemma 3.2, i.e. $\nu(r,s)=0$. So we may assume that $\nu \neq 0$. Let

$$\mathbf{N}(r,s-1) = \{i_1 = r < i_2 < \cdots < i_\nu\}. \tag{18}$$

Then

$$\lambda_{i_v} \not\equiv 0 \text{ for any } v = 1,2,\dots,\nu. \tag{19}$$

It also follows from 3.2 that

$$\mathbf{N}(r,s) \subseteq \mathbf{N}(r,s-1) \cup \{s\}. \tag{20}$$

Note that

$$\mathcal{B}(i_t,m) \equiv \mathcal{B}(i_u,m) \text{ for any } t<u \text{ and } m \geq i_u; \tag{21}$$

$$\mathcal{B}(i_t,i_u) \not\equiv 0 \text{ for any } t<u. \tag{22}$$

Indeed, $\mathcal{B}(i_t,m) \equiv \mathcal{C}(i_1,i_t) + \mathcal{B}(i_t,m) = \mathcal{B}(i_1,m)$. Similarly, $\mathcal{B}(i_u,m) \equiv \mathcal{B}(i_1,m)$. This proves (21). Now take $m = i_u$ in (21). We have $\mathcal{B}(i_t,i_u) \equiv \mathcal{B}(i_u,i_u) = \lambda_{i_u} \not\equiv 0$ by (19).

Assume first that $\lambda_s \equiv 0$. Then $s$ is not normal in $[r,s]$. So, taking into account (20), it suffices only to prove that $i_1, i_2, \dots, i_\nu$ are all normal in $[r,s]$. Let $t \in [1,\nu]$. Since $\lambda_{i_t} \not\equiv 0$ by (19), we have only to show that

$$M = \{m \in (i_t, s] \mid \mathcal{B}(i_t, m) \equiv 0\}$$

is compensated in $(i_t, s]$. Let

$$M' = \{m \in (i_t, s-1] \mid \mathcal{B}(i_t, m) \equiv 0\}.$$

Then $M'$ is compensated in $(r, s-1]$ since $i_t$ is normal in $[r, s-1]$. If $\mathcal{B}(i_t, s) \not\equiv 0$ then $M = M'$. Hence $M$ is compensated in $(i_t, s]$. If $\mathcal{B}(i_t, s) \equiv 0$ then $M = M' \cup \{s\}$. Let $Z$ be a set which compensates $M'$ in $(i_t, s-1]$. Then $Z \cup \{s\}$ compensates $M = M' \cup \{s\}$ since $\mathcal{B}(s,s) = \lambda_s \equiv 0$.

Now let $\lambda_s \not\equiv 0$. Then $s$ is normal in $[r,s]$.

Assume $\mathcal{B}(r,s) \not\equiv 0$. Since $r = i_1$ we have from (21) that $\mathcal{B}(i_t, s) \not\equiv 0$ for all $t = 1, 2, \dots, \nu$. Now as above all $i_1, i_2, \dots, i_\nu$ are normal in $[r,s]$. Taking into account (20), we get

$$\mathbf{N}(r,s) = \begin{cases} \{i_1, i_2, \dots, i_\nu\} & \text{if } \mathcal{C}(r,s) \not\equiv 0; \\ \{i_1, i_2, \dots, i_\nu\} \cup \{s\} & \text{if } \mathcal{C}(r,s) \equiv 0. \end{cases}$$

Finally, assume $\mathcal{B}(r,s) \equiv 0$. Then $\mathcal{C}(r,s) = \mathcal{B}(r,s) - \lambda_s \equiv -\lambda_s \not\equiv 0$. So, in view of (20), it suffices to prove that $i_1, i_2, \dots, i_{\nu-1}$ are normal in $[r,s]$ and $i_\nu$ is not.

We have $i = r_1$ So (20) implies $\mathcal{B}(i_t, s) \equiv 0$ for all $t = 1, 2, \dots, \nu$. Therefore

$$\begin{aligned} &\{m \in (i_t, s] \mid \mathcal{B}(i_t, m) \equiv 0\} = \{m \in (i_t, s-1] \mid \mathcal{B}(i_t, m) \equiv 0\} \cup \{s\}, \\ &t = 1, 2, \dots, \nu. \end{aligned} \tag{23}$$

Put

$$M_t = \{m \in (i_t, i_{t+1} - 1] \mid \mathcal{B}(i_t, m) \equiv 0\},\ t = 1, 2, \dots, \nu$$

($i_{\nu+1}$ is interpreted as $s$). By Lemma 3.2, $i_t$ is normal in $[i_t, i_{t+1} - 1]$ so $M_t$ is compensated in $(i_t, i_{t+1} - 1]$ by some set $Z_t \subseteq (i_t, i_{t+1} - 1]$, $t = 1, 2, \dots, \nu$. Now, (21) and (22) imply

$$\{m \in (i_t, s] \mid \mathcal{B}(i_t, m) \equiv 0\} = M_t \sqcup M_{t+1} \sqcup \cdots \sqcup M_\nu \sqcup \{s\},\ t = 1, 2, \dots, \nu. \tag{24}$$

If $t < \nu$, this set is compensated in $(i_t, s]$ by $Z_t \sqcup Z_{t+1} \sqcup \cdots \sqcup Z_\nu \sqcup \{i_\nu\}$ since $\mathcal{B}(i_\nu, s) \equiv 0$. Therefore, taking into account (19), it follows that $i_t$ is normal in $[r,s]$, $t = 1, 2, \dots, \nu - 1$.

It remains to prove that $i_\nu$ is not normal in $[r,s]$. Let $M_\nu = \{m_1, m_2, \dots, m_q\}$. Then

$$\{m \in (i_\nu, s] \mid \mathcal{B}(i_\nu, m) \equiv 0\} = \{m_1, m_2, \dots, m_q\} \cup \{s\},$$

cf. (24). Note that $\mathcal{B}(s,s) = \lambda_s \not\equiv 0$. So if $\{m_1, m_2, \dots, m_q\} \cup \{s\}$ is compensated in $(i_\nu, s]$ then there exist elements $d_1, d_2, \dots, d_q, d_{q+1} \in (i_\nu, s-1]$, all distinct, with

$\mathcal{B}(d_{q+1}, s) \equiv 0$ and $\mathcal{B}(d_v, m_v) \equiv 0$ for $v = 1, 2, \dots, q$. However,

$$\mathcal{C}(i_\nu, d_{q+1}) = \mathcal{B}(i_\nu, s) - \mathcal{B}(d_{q+1}, s) \equiv 0,$$
$$\mathcal{C}(i_\nu, d_v) = \mathcal{B}(i_\nu, m_v) - \mathcal{B}(d_v, m_v) \equiv 0, \ v = 1, 2, \dots, q.$$

So

$$\big|\{d \in (i_\nu, s-1] \mid \mathcal{C}(i_\nu, d) \equiv 0\}\big| > \big|\{m \in (i_\nu, s-1] \mid \mathcal{B}(i_\nu, m) \equiv 0\}\big|.$$

By Lemma 3.4, there exists $d \in (i_\nu, s-1]$ which is normal in $[i_\nu, s-1]$, and such that $\mathcal{C}(i_\nu, d) \equiv 0$. Then $d$ is normal in $[r, s-1]$, and $\mathcal{C}(r, d) \equiv \mathcal{C}(r, i_\nu) + \mathcal{C}(i_\nu, d) \equiv 0$, i.e. $d \in \mathbf{N}(r, s-1)$. This contradicts (18). Thus, $\{m \in (i_\nu, s] \mid \mathcal{B}(i_\nu, m) \equiv 0\}$ is not compensated in $(i_\nu, s]$. Hence $i_\nu$ is not normal in $[r, s]$. □

**Lemma 3.6.** *Let $\nu = \nu(r, s) \neq 0$,*

$$\{r = i_1 < i_2 < \cdots < i_\nu\} = \boldsymbol{N}(r, s),$$

*and $1 \leq j \leq \nu$, $i_j \leq i_{j+1} - 2$. Then $\boldsymbol{N}(i_j, i_{j+1} - 2) = \{i_j\}$ (here $i_{\nu+1}$ is interpreted as $s + 2$).*

*Proof.* Since $i_j \in \mathbf{N}(r, s)$ then $i_j \in \mathbf{N}(i_j, s)$. So, by Lemma 3.2, $i_j \in \mathbf{N}(i_j, i_{j+1} - 2)$. Assume there is $i \in (i_j, i_{j+1} - 2]$ such that $i$ is normal in $(i_j, i_{j+1} - 2]$ and $\mathcal{C}(i_j, i) \equiv 0$. We have $\mathcal{C}(r, i) = \mathcal{C}(r, i_j) + \mathcal{C}(i_j, i)$. Hence

$$\mathcal{C}(r, i) \equiv 0. \tag{25}$$

We shall prove that $i$ is normal in $[r, s]$. Then (25) will imply $i \in \mathbf{N}(r, s)$ giving a contradiction.

In view of (25), we have

$$\mathcal{B}(i, i_{j+1} - 1) \equiv \mathcal{C}(r, i) + \mathcal{B}(i, i_{j+1} - 1) = \mathcal{B}(r, i_{j+1} - 1) = \mathcal{C}(r, i_{j+1}) - 1 \equiv -1 \not\equiv 0.$$

So

$$i_{j+1} - 1 \notin \{m \in (i, s] \mid \mathcal{B}(i, m) \equiv 0\}. \tag{26}$$

Moreover, for $v \in [j + 1, \nu]$ we similarly obtain

$$\mathcal{B}(i, i_v) \equiv \mathcal{C}(r, i) + \mathcal{B}(i, i_v) = \mathcal{B}(r, i_v) = \mathcal{C}(r, i_v) + \lambda_{i_v} \equiv \lambda_{i_v}.$$

Since $i_v$ is normal in $[r, s]$ we have $\lambda_{i_v} \not\equiv 0$, see Definition 1.16. Thus

$$i_v \notin \{m \in (i, s] \mid \mathcal{B}(i, m) \equiv 0\}, \quad j + 1 \leq v \leq \nu. \tag{27}$$

Put

$$M_v = \{m \in (i_v, i_{v+1} - 1] \mid \mathcal{B}(i_v, m) \equiv 0\}, \quad j + 1 \leq v \leq \nu$$

($i_{\nu+1}$ is interpreted as $s + 2$). Let $m \in (i_v, i_{v+1} - 1]$, $j + 1 \leq v \leq \nu$. Then

$$\mathcal{C}(i, i_v) = \mathcal{C}(r, i_v) - \mathcal{C}(r, i) \equiv 0$$

by (25). Therefore

$$\mathcal{B}(i, m) = \mathcal{C}(i, i_v) + \mathcal{B}(i_v, m) \equiv (i_v, m).$$

So it follows from (26) and (27) that

$$\begin{aligned} M :&= \{m \in (i,s] \mid \mathcal{B}(i,m) \equiv 0\} \\ &= \{m \in (i, i_{j+1}-2] \mid \mathcal{B}(i,m) \equiv 0\} \sqcup M_{j+1} \sqcup \cdots \sqcup M_\nu. \end{aligned} \tag{28}$$

Since $i$ is normal in $[i_j, i_{j+1}-2]$ there exists a subset $Z \subseteq (i, i_{j+1}-2]$ which compensates $\{m \in (i, i_{j+1}-2] \mid \mathcal{B}(i,m) \equiv 0\}$. It follows from Lemma 3.2 that $i_v$ is normal in $[i_v, i_{v+1}-1]$, $j+1 \le v \le \nu$. So for such $v$ there exists $Z_v \subseteq (i_v, i_{v+1}-1]$ which compensates $M_v$ in $(i_v, i_{v+1}-1]$. Therefore $Z \sqcup Z_{j+1} \sqcup \cdots \sqcup Z_\nu$ compensates $M$. Moreover, $\lambda_i \not\equiv 0$ since $i$ is normal in $[i_j, i_{j+1}-1]$. So $i$ is normal in $[r,s]$. □

**Lemma 3.7.** *Let $\nu = \nu(r,s) > 1$ and*

$$\{r = i_1 < i_2 < \cdots < i_\nu\} = \boldsymbol{N}(r,s).$$

*Then $\nu'(i_j, i_{j+1}-1) = 0$ for all $j = 1, 2, \ldots, \nu-1$.*

*Proof.* Let $j \in [1, \nu-1]$. Put $\lambda' = \lambda(i_{j+1}-1)$, cf. (5). Then

$$\mathcal{B}_{\lambda'}(i_j, i_{j+1}-1) \equiv 0. \tag{29}$$

Indeed, we have

$$\begin{aligned} \mathcal{B}_{\lambda'}(i_j, i_{j+1}-1) &= \sum_{v=i_j}^{i_{j+1}-2} \lambda_v + (\lambda_{i_{j+1}-1}+1) + (i_{j+1}-1) - i_j \\ &= \sum_{v=i_j}^{i_{j+1}-1} \lambda_v + i_{j+1} - i_j = \mathcal{C}_\lambda(i_j, i_{j+1}). \end{aligned}$$

On the other hand,

$$\mathcal{C}_\lambda(i_j, i_{j+1}) = \mathcal{C}_\lambda(r, i_{j+1}) - \mathcal{C}_\lambda(r, i_j) \equiv 0.$$

If $i_j = i_{j+1}-1$ then $\mathcal{B}_{\lambda'}(i_j, i_{j+1}-1) = \mathcal{B}_{\lambda'}(i_j, i_j) = \lambda_{i_j}+1$. So (29) implies $\lambda_{i_j}+1 \equiv 0$. Hence $i_j$ is not normal in $[i_j, i_j]$ for $\lambda'$. Therefore $\nu'(i_j, i_{j+1}-1) = \nu'(i_j, i_j) = 0$.

Assume that $i_j \le i_{j+1}-2$. By Lemma 3.6 we have $\mathbf{N}(i_j, i_{j+1}-2) = i_j$, whence $\nu(i_j, i_{j+1}-2) = 1$. Now by virtue of Lemma 3.5 and (29) we have $\nu'(i_j, i_{j+1}-1) = \nu(i_j, i_{j+1}-2) - 1 = 0$. □

## 4. $\sigma$-subsets

In 1.16 we have defined integers $\sigma(i,j) = \sigma_\lambda(i,j)$ for any $i,j$ with $1 \le i,j \le l$, $j \ge i-1$. Recall that $\sigma(i,j) \in \{0,1\}$.

We also define

$$\sigma'(i,j) = \sigma_{\lambda(j)}(i,j); \tag{30}$$

$$\bar\sigma(i,j) = \sigma_{\lambda(i)}(i,j). \tag{31}$$

Besides essential subsets we shall need one more class of subsets called $\sigma$-subsets. It will turn out that the number of essential subsets in $(r,s]$ is equal to the number of $\sigma$-subsets in $(r,s]$.

**Definition 4.1.**

(i) Let $J=\{j_1<j_2<\dots<j_f\}\subseteq(r,s]$. We set

$$\sigma(J)=\sigma_{r,s}(J)=\sigma(r,j_1-2)\sigma(j_1,j_2-2)\dots\sigma(j_{f-1},j_f-2)\sigma(j_f,s),$$

if $J\neq\emptyset$, and

$$\sigma(\emptyset)=\sigma_{r,s}(\emptyset)=\sigma(r,s).$$

(ii) A subset $J\subseteq(r,s]$ is called a *$\sigma$-subset* (in $(r,s]$ for $\lambda$) if and only if $\sigma(J)\neq0$.

*Remark.* Clearly $\sigma(i,j)=1$ if and only if $\nu(i,j)>0$, and $\sigma(i,j)=0$ if and only if $\nu(i,j)=0$, cf. Remark after Definition 3.1.

**Lemma 4.2.** $\sharp\{\sigma\text{-}subsets\ in\ (r,s]\}=\sum_{J\subseteq(r,s]}\sigma(J)$.

*Proof.* The lemma follows immediately from Definition 4.1 since $\sigma(J)\neq0$ if and only if $\sigma(J)=1$. $\square$

**Lemma 4.3.** *Let*

$$J=\{j_1<j_2<\dots<j_f\}\subseteq(r,s]$$

*be a non-empty $\sigma$-subset. Put $j_0=r$. Assume $\nu'(j_v,j_{v+1}-1)=0$ for $v=0,1,\dots,f-1$. Then $\nu(r,s)\geq f+1$, and $r<j_1<j_2<\dots<j_f$ are the minimal $f+1$ elements of the set $\boldsymbol{N}(r,s)$.*

*Proof.* Throughout the proof we interpret $j_{f+1}$ as $s+2$. Note that

$$\lambda_{j_v}\not\equiv0,\quad v=0,1,\dots,f. \tag{32}$$

Indeed, $\sigma(J)=1$ implies $\sigma(j_v,j_{v+1}-2)=1$ for $v=0,1,\dots,f$. For $v=f$ we have $\sigma(j_f,s)=1$. Hence $j_f$ is normal in $[j_f,s]$. In particular, $\lambda_{j_f}\not\equiv0$. Let $v\in[0,f)$. If $j_v\leq j_{v+1}-2$ then $\sigma(j_v,j_{v+1}-2)\neq0$ similarly yields $\lambda_{j_v}\not\equiv0$. Let $j_v=j_{v+1}-1$. Then $\sigma'(j_v,j_{v+1}-1)=0$ means that $j_v$ is not normal in $[j_v,j_v]$ for $\lambda(j_v)$, i.e.

$$\lambda_{j_v}+1\equiv0\quad\text{if } v\in[0,f)\text{ and } j_{v+1}=j_v+1. \tag{33}$$

Therefore $\lambda_{j_v}\not\equiv0$ again. This proves (32).

We claim that

$$\mathcal{C}_\lambda(j_v,j_{v+1})\equiv0,\quad v=0,1,\dots,f-1. \tag{34}$$

Indeed, if $j_{v+1}=j_v+1$ then $\mathcal{C}_\lambda(j_v,j_{v+1})=\lambda_{j_v}+1\equiv0$ by (33). Assume $j_v\leq j_{v+1}-2$. We have $\nu(j_v,j_{v+1}-2)>0$, $\nu'(j_v,j_{v+1}-2)=0$. So Lemma 3.5 implies

$$\nu(j_v,j_{v+1}-2)=1\quad\text{for all } v\in[0,f)\text{ such that } j_v\leq j_{v+1}-2; \tag{35}$$

$$\lambda_{j_{v+1}-1}+1\not\equiv0\quad\text{for all } v\in[0,f)\text{ such that } j_v\leq j_{v+1}-2; \tag{36}$$

and

$$\mathcal{B}_{\lambda(j_{v+1}-1)}(j_v,j_{v+1}-1)\equiv0\quad\text{for all } v\in[0,f)\text{ such that } j_v\leq j_{v+1}-2. \tag{37}$$

Note that

$$\mathcal{B}_{\lambda(j_{v+1}-1)}(j_v, j_{v+1}-1) = \sum_{w=j_v}^{j_{v+1}-2} \lambda_w + (\lambda_{j_{v+1}-1}+1) + (j_{v+1}-1) - j_v = \mathcal{C}_\lambda(j_v, j_{v+1}).$$

Now (37) implies $\mathcal{C}(j_v, j_{v+1}) \equiv 0$ for $v \in [0, f)$ with $j_v \le j_{v+1} - 2$. This completes the proof of (34).

Since $\mathcal{C}(j_v, j_w) = \mathcal{C}(j_v, j_{v+1}) + \mathcal{C}(j_{v+1}, j_{v+2}) + \cdots + \mathcal{C}(j_{w-1}, j_w)$, (34) implies

(38) $$\mathcal{C}(j_v, j_w) \equiv 0 \quad \text{for any } 0 \le v < w \le f.$$

Put

$$M_v = \{m \in (j_v, j_{v+1}-2] \mid \mathcal{B}(j_v, m) \equiv 0\}, \quad v = 0, 1, \ldots, f.$$

Of course, $M_v = \emptyset$ if $j_{v+1} = j_v + 1$.

Assume $j_v \le j_{v+1} - 2$. Then $\sigma(j_v, j_{v+1}-2) = 1$ means that $j_v$ is normal in $[j_v, j_{v+1}-2]$. Hence there exists $Z_v \subseteq (j_v, j_{v+1}-2]$ which compensates $M_v$ in $(j_v, j_{v+1}-2]$. If $j_v - j_{v+1} - 1$ we set $Z_v = \emptyset$. We claim that

(39) $$\{m \in (j_v, s] \mid \mathcal{B}_\lambda(j_v, m) \equiv 0\} = M_v \sqcup M_{v+1} \sqcup \cdots \sqcup M_f, \quad v = 0, 1, \ldots, f.$$

Indeed, using (38) we get for any $w \in (v, f]$:

$$\mathcal{B}_\lambda(j_v, j_w - 1) = \mathcal{C}_\lambda(j_v, j_w) - 1 \equiv -1 \not\equiv 0,$$
$$\mathcal{B}(j_v, j_w) = \mathcal{C}_\lambda(j_v, j_w) + \lambda_{j_w} \equiv \lambda_{j_w} \not\equiv 0,$$

where we have used (32) for the last inequality. Hence

$$\{m \in (j_v, s] \mid \mathcal{B}_\lambda(j_v, m) \equiv 0\} \subseteq \cup_{w=v}^{f} (j_w, j_{w+1}-2].$$

Let $m \in (j_w, j_{w+1}-2]$ for some $w \in [v, f]$. Then, using (38) again, we have

$$\mathcal{B}_\lambda(j_v, m) = \mathcal{C}_\lambda(j_v, j_w) + \mathcal{B}_\lambda(j_w, m) \equiv \mathcal{B}_\lambda(j_w, m).$$

This proves (39).

However, $Z_v \sqcup Z_{v+1} \sqcup \cdots \sqcup Z_f$ clearly compensates $M_v \sqcup M_{v+1} \sqcup \cdots \sqcup M_f$. So, taking into account (32) we conclude that $r = j_0, j_1, \ldots, j_f$ are normal in $[r, s]$. In view of (38), $\mathcal{C}(r, j_v) \equiv 0$ for $v = 0, 1, \ldots, f$. So, moreover, $j_0, j_1, \ldots, j_f \in \mathbf{N}(r, s)$.

It remains to prove that $i \in (r, j_f) \setminus \{j_1, j_2, \ldots, j_{f-1}\}$ implies $i \notin \mathbf{N}(r, s)$. Assume not. We have

(40) $$\mathcal{C}(r, i) \equiv 0.$$

Suppose first that $i \in (j_v, j_{v+1}-2]$. Then $j_v \le j_{v+1} - 2$. So (35) implies $\mathbf{N}(j_v, j_{v+1}-2) = \{j_v\}$. In view of (38), we have

$$\mathcal{C}(j_v, i) \equiv \mathcal{C}(r, j_v) + \mathcal{C}(j_v, i) = \mathcal{C}(r, i) \equiv 0,$$

see (40). So if $i$ is normal in $[j_v, j_{v+1}-2]$ we would have $i \in \mathbf{N}(j_v, j_{v+1}-2)$ giving a contradiction. Thus, $i$ is not normal in $[j_v, j_{v+1}-2]$. By Lemma 3.2, $i$ is not normal in $[r, s]$. This contradicts the assumption $i \in \mathbf{N}(r, s)$.

Now suppose $i = j_{v+1} - 1$. Then $j_v \leq j_{v+1} - 2$. The equation (40) implies $\mathcal{C}_\lambda(r, j_{v+1} - 1) \equiv 0$. This together with $\mathcal{C}_\lambda(r, j_v) \equiv 0$ yields $\lambda_{j_{v+1}-1} + 1 \equiv 0$. However this contradicts (36). $\square$

Recall the constants $\nu(i,j), \nu'(i,j)$ introduced in 3.1.

**Definition 4.4.** Let $J = \{j_1 < j_2 < \cdots < j_f\} \subseteq (r, s]$. We set

$$\nu(J) = \nu_{r,s}(J) = \nu'(r, j_1 - 1)\nu'(j_1, j_2 - 1)\dots\nu'(j_{f-1}, j_f - 1)\nu(j_f, s),$$

if $J \neq \emptyset$, and

$$\nu(\emptyset) = \nu_{r,s}(\emptyset) = \nu(r, s).$$

**Lemma 4.5.** $\sharp\{\sigma$*-subsets in* $(r, s]\} = \sum_{J \subseteq (r,s]} \nu(J)$.

*Proof. Step 1.* We associate to any $\sigma$-subset $M$ a subset $\varphi(M)$. We shall prove in Step 3 that $\nu(\varphi(M)) \neq 0$.

Let $M = \{m_1 < m_2 < \cdots < m_h\}$ be a $\sigma$-subset (in $(r, s]$ for $\lambda$). We interpret $m_0$ as $r$. Put

$$\begin{aligned} \varphi(M) &= \{m_{w_1} < m_{w_2} < \cdots < m_{w_f}\} \\ &= \{m_w \mid 1 \leq w \leq h,\ \nu'(m_{w-1}, m_w - 1) \neq 0\}. \end{aligned} \tag{41}$$

We interpret $w_0$ as 0.

*Step 2.* To any subset $J$ with $\nu(J) \neq 0$ we associate $\nu(J)$ subsets $J(x_0, x_1, \dots, x_f)$. We shall prove in Step 4 that they are $\sigma$-subsets.

Let $J = \{j_1 < j_2 < \cdots < j_f\} \subseteq (r, s]$ be a subset in $(r, s]$ with $\nu(J) \neq 0$. We interpret $j_0$ as $r$ and $j_{f+1}$ as $s + 2$.

Put $\nu_v = \nu'(j_v, j_{v+1} - 1)$, $v = 0, 1, \dots, f - 1$; $\nu_f = \nu(j_f, s)$. Then $\nu(J) = \nu_0\nu_1\dots\nu_f$. We define $\nu(J)$ subsets $J(x_0, x_1, \dots, x_f)$; $1 \leq x_v \leq \nu_v$, $v = 0, 1, \dots, f$, as follows. Let

$$\left\{j_v = i_1^{(v)} < i_2^{(v)} < \cdots < i_{\nu_v}^{(v)}\right\} = \mathbf{N}'(j_v, j_{v+1} - 1), \quad v = 0, 1, \dots, f - 1, \tag{42}$$

$$\left\{j_f = i_1^{(f)} < i_2^{(f)} < \cdots < i_{\nu_f}^{(f)}\right\} = \mathbf{N}(j_f, s) \tag{43}$$

(cf. 3.1 for the definition of $\mathbf{N}'(i, j)$).

Now put

$$\begin{aligned} &J(x_0, x_1, \dots, x_f) = J \cup \left(\cup_{v=0}^{f} \cup_{q_v=2}^{x_v} \{i_{q_v}^{(v)}\}\right), \\ &1 \leq x_v \leq \nu_v,\ v = 0, 1, \dots, f. \end{aligned} \tag{44}$$

*Step 3.* We show that

$$\begin{aligned} &\nu(\varphi(M)) \neq 0, \text{ and if we put } \varphi(M) = \{j_1 < j_2 < \cdots < j_f\} = J \quad \text{then} \\ &\text{there are some values } x_0, x_1, \dots, x_f \text{ such that } M = J(x_0, x_1, \dots, x_f). \end{aligned} \tag{45}$$

To prove (45) we first show that

$$(46)\qquad \begin{aligned} &\nu'(m_{w_v}, m_{w_{v+1}}-1) \ge w_{v+1}-w_v, \quad \text{and} \\ &m_{w_v} < m_{w_v+1} < \cdots < m_{w_{v+1}-1} \quad \text{are the minimal } w_{v+1}-w_v \\ &\text{elements of the set } \mathbf{N}'(m_{w_v}, m_{w_{v+1}}-1); \quad v = 0,1,\dots,f-1. \end{aligned}$$

Indeed, if $w_v = w_{v+1}-1$ then $\nu'(m_{w_v}, m_{w_{v+1}}-1) \neq 0$ by definition, cf. (41). So (46) is evident. Now assume $w_v < w_{v+1}-1$. We want to apply Lemma 4.3 to

$$\begin{aligned} \{m_{w_v}, m_{w_{v+1}}-1\} \quad &\text{instead of} \quad \{r,s\}, \\ \lambda' = \lambda(m_{w_{v+1}}-1) \quad &\text{instead of} \quad \lambda, \quad \text{and} \\ M^v = \{m_{w_v+1} < m_{w_v+2} < \cdots < m_{w_{v+1}-1}\} \quad &\text{instead of} \quad J = \{j_1 < j_2 < \cdots < j_f\}. \end{aligned}$$

We may do it because

(a) $M^v \neq \emptyset$ (since $w_v < w_{v+1}-1$).

(b) $M^v$ is a $\sigma$-subset in $(m_{w_v}, m_{w_{v+1}}-1]$ for $\lambda'$. Indeed, $\sigma(m_i, m_{i+1}-2) \neq 0$ for $i \in [w_v, w_{v+1}-1)$ since $M$ is a $\sigma$-subset in $(r,s]$. And it remains to note that $\sigma'(m_{w_{v+1}-1}, m_{w_{v+1}}-1) \neq 0$ by the definition of $w_j$, cf. (41).

(c) $\nu'(m_i, m_{i+1}-1) = 0$ for $i \in [w_v, w_{v+1}-1)$ (by the definition of $w_j$, cf. (41)).

Now Lemma 4.3 implies (46).

Similarly one proves

$$(47)\qquad \begin{aligned} &\nu(m_{w_f}, s) \ge h - w_f + 1, \quad \text{and} \quad m_{w_f} < m_{w_f+1} < \cdots < m_h \\ &\text{are the } h - w_f + 1 \text{ minimal elements of the set } \mathbf{N}(m_{w_f}, s). \end{aligned}$$

Now (45) follows from (46) and (47).

*Step 4.* Let $J = \{j_1 < j_2 < \cdots < j_f \subseteq (r,s]$, $\nu(J) \neq 0$. We show that

$$(48)\qquad J = J(x_0, x_1, \dots, x_f) \quad \text{is a } \sigma\text{-set for all admissible } x_0, x_1, \dots, x_f.$$

For, we have to prove that $\sigma(i_u^{(v)}, i_{u+1}^{(v)}-2) = 1$ for any $u = 1,2,\dots,x_v-1$, and $\sigma(i_{x_v}^{(v)}, j_{v+1}-2) = 1$; $v = 0,1,\dots,f$ (cf. (42), (43)).

Let $v \in [0,f)$, and $u \in [1,x_v)$. Since $\sigma(j,j-1) = 1$ for any integer $j$ we may assume that $i_u^{(v)} \le i_{u+1}^{(v)}-2$. Since $i_u^{(v)} \in \mathbf{N}'(j_v, j_{v+1}-1)$ then $i_u^{(v)}$ is normal in $[i_u^{(v)}, i_{u+1}^{(v)}-2]$ by Lemma 3.2. Therefore $\sigma(i_u^{(v)}, i_{u+1}^{(v)}-2) = 1$. The case $v = f$, $u \in [1,x_f)$ is considered similarly.

Let $v \in [0,f)$. We may assume that $i_{x_v}^{(v)} \le j_{v+1}-2$. Since $i_{x_v}^{(v)} \in \mathbf{N}'(i_1^{(v)}, j_{v+1}-1)$ we have $i_{x_v}^{(v)}$ is normal in $[i_{x_v}^{(v)}, j_{v+1}-2]$ by Lemma 3.2, i.e. $\sigma(i_u^{(v)}, i_{u+1}^{(v)}-2) = 1$. The case $v = f$ is considered similarly. Thus, (48) is proved.

*Step 5.* Let $J = \{j_1 < j_2 < \cdots < j_f \subseteq (r,s]$, $\nu(J) \neq 0$. We show that

$$(49)\qquad \varphi(J(x_0, x_1, \dots, x_f)) = J.$$

For, we have to prove that

(a) $\nu'(i_u^{(v)}, i_{u+1}^{(v)}-1) = 0$ for all $u \in [1,x_f)$, $v = 0,1,\dots,f$ and

(b) $\nu'(i^{(v)}_{x_v}, j_{v+1}-1) \neq 0$, $v = 0, 1, \dots, f-1$.

Since $i^{(v)}_{x_v} \in \mathbf{N}'(j_v, j_{v+1}-1)$ (cf. (42), (43)) we have $i^{(v)}_{x_v}$ is normal in $[j_v, j_{v+1}-1]$ for $\lambda(j_{v+1}-1)$, i.e. $\nu'(i^{(v)}_{x_v}, j_{v+1}-1) \neq 0$, giving (b). To prove (a) we apply Lemma 3.7 with

$$\begin{aligned} \{j_v, j_{v+1}-1\} &\quad \text{instead of} \quad \{r, s\}, \\ \lambda(j_{v+1}-1) &\quad \text{instead of} \quad \lambda, \quad \text{and} \\ i^{(v)}_1 < i^{(v)}_2 < \dots < i^{(v)}_{\nu_v} &\quad \text{instead of} \quad i_1 < i_2 < \dots < i_\nu. \end{aligned}$$

The equality (49) is proved.

Thus, to every subset $J \subseteq (r, s]$ with $\nu(J) \neq 0$ we have associated $\nu(J)$ distinct $\sigma$-subsets $J(x_0, x_1, \dots, x_f)$, cf (44), (48). In view of (49), we also get that $J(x_0, x_1, \dots, x_f) \neq J'(x'_0, x'_1, \dots, x'_f)$ if $J \neq J'$. On the other hand, any $\sigma$-subset has the form $J(x_0, x_1, \dots, x_f)$ for some set $J$ with $\nu(J) \neq 0$, cf. (45). This proves the lemma. $\square$

## 5. Recurrence formulas for the number of essential subsets

**Definition 5.1.** Let $1 \leq t, u \leq l$, $J \subseteq (t, u]$. We call $J$ *regular* (in $(t, u]$ for $\lambda$) if and only if there exists a weakly increasing injection $d : J \to (t, u]$ such that $\mathcal{B}(j, d(j)) \equiv 0$ for all $j \in J$.

*Remark.* (i) The empty set is regular by definition.

(ii) We do not assume $t \leq u$ in 5.1 since we shall need to consider formally $\emptyset$ as a regular subset in $(r, r-1]$ (in the proof of Proposition 5.5 below).

**Lemma 5.2.** *Let $J$ be a regular subset in $(r, s]$. Then for any $j \in J$ the set $M \cap (j, s]$ is non-essential in $(j, s]$.*

*Proof.* This follows immediately from Definitions 5.1 and 1.13. $\square$

We now prove a technical lemma which is only used to deduce Corollary 5.4.

**Lemma 5.3.** *Let $J \subseteq (r, s]$ be a regular subset. Then $J$ is essential if and only if $r$ is normal in $[r, s]$ and $J$ is maximal among the regular subsets in $(r, s]$ (with respect to "$<$", cf.* 2.5*).*

*Proof.* Let $J = \{j_1, j_2, \dots, j_f\}$. Set

$$M = \{m_1 < m_2 < \dots < m_e\} = \{m \in [r, s] \mid \mathcal{B}(r, m) \equiv 0\}.$$

Since $J$ is regular there exist $d_1, d_2, \dots, d_f$, all distinct, with $d_i \in [j_i, s]$ and $\mathcal{B}(j_i, d_i) \equiv 0$, $i = 1, 2, \dots, f$. Put

$$D = \{d_1, d_2, \dots, d_f\}.$$

Assume $J$ is essential. Then $D \supseteq M$. Indeed, if $m \in M$, $m \notin D$ then we can take $t = r$, $u = s + 1$, $d(j_i) = d_i$, $i = 1, 2, \dots, f$, $d(t) = m$ in Definition 1.13, i.e. $J$ is not essential.

Now $r \notin D$ implies $r \notin M$, i.e. $\mathcal{B}(r,r) \not\equiv 0$. Since $\mathcal{B}(r,r) = \lambda_r$ we get

$$\lambda_r \not\equiv 0. \tag{50}$$

Let $m_i = d_{v_i}$, $i = 1, 2, \ldots, e$. Then $\{j_{v_1}, j_{v_2}, \ldots, j_{v_e}\}$ compensates $M$. Together with (50) this means that $r$ is normal in $[r,s]$, cf. 1.16.

Next we prove that $J$ is maximal among the regular subsets in $(r,s]$. We may assume that $j_f < j_{f-1} < \cdots < j_1$. Let

$$I = \{i_g < i_{g-1} < \cdots < i_1\} \subseteq (r,s]$$

be another regular subset. Assume $I > J$. Then, in view of 2.5, one of the following conditions holds:

(a) there exists $c \in [1, \min(f,g)]$ such that $i_v = j_v$ for $v \in [1,c)$, and $i_c > j_c$;
(b) $g > f$ and $i_v = j_v$ for $v \in [1,f]$.

Assume (a) holds. Since $I$ is regular, there exist $\delta_1, \delta_2, \ldots, \delta_g$, all distinct, such that $\delta_v \in [i_v, s]$ and $\mathcal{B}(i_v, \delta_v) \equiv 0$ for all $v \in [1,g]$. However $i_v = j_v$ for $v \in [1,c)$. So $\mathcal{B}(j_v, \delta_v) \equiv 0$, $v = 1, 2, \ldots, c-1$. Moreover, $\{j_1, j_2, \ldots, j_{c-1}\} = J \cap (i_c, s]$, and $i_c \notin J$. So we may take $t = i_c$, $u = s+1$, $d(j_v) = \delta_v$ for $v \in [1,c)$ and $d(t) = \delta_c$ in Definition 1.13. Thus, $J$ is not essential, giving a contradiction.

Case (b) is considered similarly.

In the other direction, let $r$ be normal in $[r,s]$ and $J$ be maximal among the regular subsets in $(r,s]$. Suppose $J$ is not essential.

Now it is convenient to assume that $j_1 < j_2 < \cdots < j_f$.

By Definition 1.13 there exist $t, u \in [r, s+1] \setminus J$, with $t < u$, and a weakly increasing injection $\delta : \{t\} \cup ((t,u) \cap J) \to [t,u)$, such that $\mathcal{B}(k, \delta(k)) \equiv 0$ for all $k \in \{t\} \cup ((t,u) \cap J)$.

Assume first that $t > r$. Let

$$(t,u) \cap J = \{j_a < j_{a+1} < \cdots < j_b\}.$$

Note that $\{t\} \cup (J \cap (t,s]) > J$ since $t \notin J$.

Moreover, we claim that $\{t\} \cup (J \cap (t,s])$ is regular in $(r,s]$. Indeed, notice that

$$\{t\} \cup (J \cap (t,s]) = t, j_a, j_{a+1}, \ldots, j_b, j_{b+1}, \ldots, j_f.$$

Besides, $\mathcal{B}(t, \delta(t)) \equiv 0$, $\mathcal{B}(j_v, \delta(j_v)) \equiv 0$ for $v \in [a,b]$, and $\mathcal{B}(j_v, d_v) \equiv 0$ for $v \in (b,f]$. It remains to observe that

$$\delta(t), \delta(j_a), \delta(j_{a+1}), \ldots, \delta(j_b), d_{b+1}, \ldots, d_f$$

are all distinct since $\delta(t) < u$ and $\delta(j_v) < u$ for $v \in [a,b]$, while $d_v \geq j_v > u$ for $v \in (b,f]$.

Thus, we have proved that $\{t\} \cup (J \cap (t,s]) > J$ and $\{t\} \cup (J \cap (t,s])$ is regular. This contradicts the assumptions on $J$.

Now assume that $t = r$. Put

$$\begin{aligned}
J' &= \{j_1 < j_2 < \cdots < j_b\} = J \cap (r, u),\\
\Delta &= \{\delta(j_1), \delta(j_2), \dots, \delta(j_b)\},\\
M &= \{m_1 < m_2 < \cdots < m_h\} = \{m \in (r, u) \mid \mathcal{B}(r, m) \equiv 0\}.
\end{aligned}$$

Since $r$ is normal in $[r, s]$ there exist $z(m_1), z(m_2), \dots, z(m_h)$, all distinct, with $z(m_v) \in (r, m_v]$ and $\mathcal{B}(z(m_v), m_v) \equiv 0$, $v = 1, 2, \dots, h$. Put

$$Z = \{z(m_1), z(m_2), \dots, z(m_h)\}.$$

Note that

$$\delta : J' \to \Delta \qquad \text{and} \qquad z : M \to Z$$

are bijections.

Since $\mathcal{B}(r, \delta(t)) \equiv 0$ we have

$$\delta(t) \in M. \tag{51}$$

We claim that

$$Z \not\subseteq J'. \tag{52}$$

Indeed, let $Z \subseteq J'$. Then there exist $c_v \in [1, b]$ such that $z(m_v) = j_{c_v}$, $v = 1, 2, \dots, h$. We have

$$\mathcal{C}(r, j_{c_v}) = \mathcal{C}(r, z(m_v)) = \mathcal{B}(r, m_v) - \mathcal{B}(z(m_v), m_v) \equiv 0.$$

So

$$\mathcal{B}(r, \delta(j_{c_v})) = \mathcal{C}(r, j_{c_v}) + \mathcal{B}(j_{c_v}, \delta(j_{c_v})) \equiv 0, \ v = 1, 2, \dots, h.$$

Therefore $\delta(j_{c_1}), \delta(j_{c_2}), \dots, \delta(j_{c_h})$ are $h$ distinct elements of $M$. Since $|M| = h$ this yields $M \subseteq \Delta$, which contradicts (51) since $\delta(t) \notin \Delta$.

We define an integer $q$ and elements

$$\begin{aligned}
z^{(0)}, z^{(1)}, \dots, z^{(q)} &\in Z;\\
j^{(1)}, \dots, j^{(q)} &\in J';\\
\delta^{(1)}, \dots, \delta^{(q)} &\in \Delta;\\
m^{(0)}, m^{(1)}, \dots, m^{(q)} &\in M
\end{aligned}$$

inductively as follows. Put

$$m^{(0)} = \delta(t).$$

Then $m^{(0)} \in M$ by (51). So we may define an element $z^{(0)} \in Z$ by putting

$$z^{(0)} = z(m^{(0)}).$$

If $z^{(0)} \notin J'$ we put $q = 0$.

Assume that $i > 0$, that $z^{(v)}, m^{(v)}$ have been defined for all $v \in [0, i)$, and that $j^{(v)}, \delta^{(v)}$ have been defined for all $v \in [1, i)$. If $z^{(i-1)} \notin J'$ we put $q = i-1$. Otherwise define $z^{(i)}, j^{(i)}, \delta^{(i)}, m^{(i)}$ as follows. Put

$$j^{(i)} = z^{(i-1)}.$$

Since $z^{(i-1)} \in J'$ we have $j^{(i)} \in J'$, and so we may set

$$\delta^{(i)} = \delta(j^{(i)}).$$

Since $j^{(i)} \in Z$ we have $j^{(i)} = z(m)$ for some $m \in M$. So

$$\mathcal{C}(r, j^{(i)}) = \mathcal{C}(r, z(m)) = \mathcal{B}(r, m) - \mathcal{B}(z(m), m) \equiv 0.$$

Hence

$$\mathcal{B}(r, \delta^{(i)}) = \mathcal{B}(r, \delta(j^{(i)})) = \mathcal{C}(r, j^{(i)}) + \mathcal{B}(j^{(i)}, \delta(j^{(i)})) \equiv 0.$$

Thus, $\delta^{(i)} \in M$ since $\delta^{(i)} \in (r, u)$. So we may set

$$m^{(i)} = \delta^{(i)}.$$

Finally put

$$z^{(i)} = z(m^{(i)}).$$

Recall that $Z \not\subseteq J'$ by (52). So to show that $q$ is well defined, it suffices to prove that $z^{(0)}, z^{(1)}, \ldots, z^{(i)}$ are all distinct. If $i = 0$ this is trivial. Let $i > 0$, and $z^{(0)}, z^{(1)}, \ldots, z^{(i-1)}$ be all distinct. Since $z^{(v)} = z(m^{(v)})$, and $z$ is bijective, it suffices to prove that $m^{(0)}, m^{(1)}, \ldots, m^{(i)}$ are all distinct. Recall that $m^{(0)} = \delta(t)$, $m^{(v)} = \delta^{(v)}$ for $v \in [1, i]$. So we have to prove that $\delta(t), \delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(i)}$ are all distinct. However $\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(i)} \in \Delta$, while $\delta(t) \notin \Delta$. So it is enough to check that $\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(i)}$ are all distinct. But $\delta^{(v)} = \delta(j^{(v)})$, $v = 1, 2, \ldots, i$, and $\delta$ is bijective. So we need only to show that $j^{(1)}, j^{(2)}, \ldots, j^{(i)}$ are all distinct. Now it suffices to use $j^{(v)} = z^{(v-1)}$, $v = 1, 2, \ldots, i$, and the inductive hypothesis.

Let us consider the set $I = \{z^{(q)}\} \cup J$. By definition of $q$ we have $z^{(q)} \notin J'$. Since $z^{(q)} \in (r, u)$, and $J' = J \cap (r, u)$, we conclude that $z^{(q)} \notin J$. Hence $I > J$.

We claim that $I$ is regular. Indeed, $\delta^{(q)} = m^{(q)}$, $z^{(q)} = z(m^{(q)})$. So

$$z^{(q)} \leq \delta^{(q)} \quad \text{and} \quad \mathcal{B}(z^{(q)}, \delta^{(q)}) \equiv 0. \tag{53}$$

Let $v \in [2, q]$. Then $j^{(v)} = z^{(v-1)} = z(m^{(v-1)})$, $m^{(v-1)} = \delta^{(v-1)}$. So

$$j^{(v)} \leq \delta^{(v-1)} \quad \text{and} \quad \mathcal{B}(j^{(v)}, \delta^{(v-1)}) \equiv 0 \quad \text{for} \quad v = 2, 3, \ldots, q. \tag{54}$$

Moreover, $j^{(1)} = z^{(0)} = z(m^{(0)})$, $m^{(0)} = \delta(t)$. So

$$j^{(1)} \leq \delta(t) \quad \text{and} \quad \mathcal{B}(j^{(1)}, \delta(t)) \equiv 0. \tag{55}$$

We also have

$$\mathcal{B}(j, \delta(j)) \equiv 0 \quad \text{for} \quad j \in J' \setminus \{j^{(1)}, j^{(2)}, \ldots, j^{(q)}\}, \tag{56}$$

and

$$\mathcal{B}(j_v, d_v) \equiv 0 \quad \text{for} \quad v \in (b, f]. \tag{57}$$

In view of (53)-(57), to prove that $I$ is regular it suffices to check that

$$\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(q)},\ \delta(t),\ \delta(j) \text{ for } j \in J' \setminus \{j^{(1)}, j^{(2)}, \dots, j^{(q)}\},\ d_{b+1}, d_{b+2}, \dots, d_f$$

are all distinct. Note that $d_{b+1}, d_{b+2}, \dots, d_f > u$. On the other hand, $\delta^{(v)} < u$, $v = 1, 2, \dots, q$; $\delta(t) < u$; and $\delta(j) < u$ for $j \in J' \setminus \{j^{(1)}, j^{(2)}, \dots, j^{(q)}\}$. So it is enough to verify that

$$\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(q)},\ \delta(t),\ \delta(j) \text{ for } j \in J' \setminus \{j^{(1)}, j^{(2)}, \dots, j^{(q)}\}$$

are all distinct. But this follows immediately from the following facts:

(a) $\delta^{(v)} = \delta(j^{(v)}),\ v = 1, 2, \dots, q$;
(b) $\delta : J' \to \Delta$ is bijective;
(c) $\delta(t) \notin \Delta$.

Thus, we have found a regular set $I > J$. This contradicts our assumption. $\square$

**Corollary 5.4.** *There exists an essential regular subset in $(r, s]$ if and only if $\sigma(r, s) = 1$. In this case such a set is unique.*

*Proof.* Recall that $\sigma(r, s) = 1$ if and only if $r$ is normal in $[r, s]$, cf. 1.13. Moreover, the set of regular subsets is always non-empty since $\emptyset$ is regular. Now the statement follows from 5.3. $\square$

**Proposition 5.5.** $\varepsilon(r, s) = \sigma(r, s) + \sum_{v=r+1}^{s} \sigma(r, v-2)\varepsilon(v, s)$.

*Proof.* Let $v \in (r, s]$, and assume that $\sigma(r, v-2) = 1$. Then by 5.4 there exists a unique essential regular subset $I \subseteq (r, v-2]$. (If $v = r+1$ we take $I = \emptyset$, cf. Remark after 5.1). Let $J \in \mathbf{E}(v, s)$. Then by Lemma 2.4, the set $I \cup \{v\} \cup J$ is essential in $(r, s]$. We denote this set by $\Lambda(v, J)$. Remark that $\Lambda(v, J)$ is defined for any $v \in (r, s]$ such that $\sigma(r, v-2) \neq 0$ and any $J \in \mathbf{E}(v, s)$. Note also that $\Lambda(v, J)$ does not depend in $I$ since $I$ is uniquely determined by $v$.

Assume $\sigma(r, s) = 1$. Then by 5.4 there exists a unique essential regular subset $I \subseteq (r, s]$. We denote it by $\Lambda_0$.

Note that we have constructed $\sigma(r, s) + \sum_{v=r+1}^{s} \sigma(r, v-2)\varepsilon(v, s)$ essential subsets in $(r, s]$. But we need to check that they are all distinct, and that any essential subset has a form $\Lambda(v, J)$ or $\Lambda_0$.

We first prove that $v$ is uniquely determined by the set $\Lambda(v, J)$ as follows.

> $(*)$ Let $v \in (r, s]$, $\sigma(r, v-2) = 1$, $J \in \mathbf{E}(v, s)$. Set $m$ to be the maximal element of $\Lambda(v, J)$ such that the following two conditions hold.
> (a) $m - 1 \notin \Lambda(v, J)$;
> (b) $\Lambda(v, J) \cap (r, m-2]$ is regular in $(r, m-2]$.
> Then $m = v$.

Indeed, $m = v$ clearly satisfies (a) and (b). Assume $m > v$. By (b), for all $x \in \Lambda(v, J) \cap (r, m-2]$ there exist $\delta(x)$, all distinct, with $x \le \delta(x) \le m-2$ and $\mathcal{B}(x, \delta(x)) \equiv 0$. Note that (a) and $m > v$ imply $v \in \Lambda(v, J) \cap (r, m-2]$. So we may take $r = v$, $s = s$, $J = J$, $t = v$, $u = m-1$ (see (a)), $d(t) = \delta(v)$, $d(x) = \delta(x)$ for

$x \in J \cap (t,u)$ in Definition 1.13. Thus $J$ is not essential in $(v,s]$. The contradiction obtained proves $(*)$.

Next, we prove

> $(**)$ All the sets $\Lambda(v,J)$ with $v \in (r,s]$, $\sigma(r,v-2)=1$, $J \in \mathbf{E}(v,s)$ are distinct. Moreover, if $\sigma(r,s)=1$ then
>
> $$\Lambda_0 \notin \{\Lambda(v,J) \mid v \in (r,s],\ \sigma(r,v-2)=1,\ J \in \mathbf{E}(v,s)\}.$$

Indeed, it is clear that $\Lambda(v,J_1) \neq \Lambda(v,J_2)$ if $J_1 \neq J_2$. It also follows from $(*)$ that $\Lambda(v_1,J_1) \neq \Lambda(v_2,J_2)$ if $v_1 \neq v_2$. Finally, let $\sigma(r,s)=1$. The set $\Lambda_0$ is regular in $(r,s]$ by definition. By Lemma 5.2, we get that $\Lambda_0 \cap (v,s]$ is not essential in $(v,s]$ for any $v \in \Lambda_0$. But $\Lambda(v,J) \cap (v,s] = J$ is essential in $(v,s]$. So $\Lambda_0 \neq \Lambda(v,J)$ for all admissible $v$ and $J$. Thus $(**)$ is proved.

It remains to prove that any essential subset in $(r,s]$ coincides with one of the $\Lambda(v,J)$ or $\Lambda_0$.

Let $M \in \mathbf{E}(r,s)$. If $M$ is regular in $(r,s]$ then $M = \Lambda_0$ by 5.4. So we may assume that $M$ is not regular. Let $v$ be the maximal element of $M$ such that

(1) $v-1 \notin M$;

(2) $M \cap (r,v-2]$ is regular in $(r,v-2]$.

Note that $v$ is well defined because (1) and (2) hold for $v = \min M$.

We claim that $M \cap (v,s]$ is essential in $(v,s]$. Assume not. By Definition 1.13, there are $t,u \in [v,s+1] \setminus (M \cap (v,s])$, with $t<u$ and a weakly increasing injection $d : \{t\} \cup ((M \cap (v,s]) \cap (t,u)) \to [t,u)$ such that $\mathcal{B}(k,d(k)) \equiv 0$ for all $k \in \{t\} \cup ((M \cap (v,s]) \cap (t,u))$. (Note that actually $(M \cap (v,s]) \cap (t,u) = M \cap (t,u)$).

If $t>v$ then $t \in (r,s] \setminus M$. Besides, $u \in (r,s+1] \setminus M$, and thus $M$ is not essential in $(r,s]$, giving a contradiction.

Let $t=v$. Then $M \cap (r,u)$ is regular in $(r,u)$, since $M \cap (r,v-2]$ is regular in $(r,v-2]$, $v-1 \notin M$, and for all $j \in M \cap [v,u)$ there exist $d(j)$, all distinct, with $j \le d(j) < u$, $\mathcal{B}(j,d(j)) \equiv 0$. If $(u,s] \cap M = \emptyset$ this implies that $M$ is regular in $(r,s]$, since $u \notin M$. So we may assume $(u,s] \cap M \neq \emptyset$. Let $m = \min\{(u,s] \cap M\}$. Then $m-1 \ge u$, so $m-1 \notin M$. We also have $M \cap (r,m-2]$ is regular in $(r,m-2]$. Now $m>u>v$ contradicts the choice of $v$.

Thus, $M \cap (v,s]$ is essential in $(v,s]$.

Moreover, $M \cap (r,v-2]$ is regular in $(r,v-2]$ by the definition of $v$, and is essential in $(r,v-2]$ by Lemma 2.2. Since $M = (M \cap (r,v-2]) \cup \{v\} \cup (M \cap (v,s])$ we get $M = \Lambda(v, M \cap (v,s])$. $\square$

**Corollary 5.6.**

(i) $\varepsilon(r,s) = \sum_{J \subseteq (r,s]} \sigma(J)$;
(ii) $\varepsilon(r,s) = \sharp\{\sigma\text{-}subsets\ in\ (r,s]\}$;
(iii) $\varepsilon(r,s) = \sum_{J \subseteq (r,s]} \nu(J)$;
(iv) $\varepsilon(r,s) = \nu(r,s) + \sum_{v=r+1}^{s} \nu'(r,v-1)\varepsilon(v,s)$.

*Proof.* By Proposition 5.5,

$$\varepsilon(r,s)=\sigma(r,s)+\sum_{v=r+1}^{s}\sigma(r,v-2)\varepsilon(v,s).$$

In particular, $\varepsilon(r,r)=\sigma(r,r)$ which gives (i) for $s=r$ since $\sigma(\emptyset)=\sigma(r,s)$. Assume (i) has been proved for all $r',s'$ with $s'-r'<s-r$. Then (see Definition 4.1)

$$\begin{aligned}\varepsilon(r,s)&=\sigma_{r,s}(\emptyset)+\sum_{v=r+1}^{s}\sigma(r,v-2)\sum_{J\subseteq(v,s]}\sigma_{v,s}(J)\\&=\sigma_{r,s}(\emptyset)+\sum_{v=r+1}^{s}\sum_{J\subseteq(r,s],J\neq\emptyset,\min J=v}\sigma_{r,s}(J)\\&=\sum_{J\subseteq(r,s]}\sigma_{r,s}(J).\end{aligned}$$

Thus (i) is proved. Part (ii) follows from (i) and Lemma 4.2. Part (iii) follows from (ii) and Lemma 4.5. To prove (iv) note that

$$\begin{aligned}\sum_{J\subseteq(r,s]}\nu(J)&=\nu_{r,s}(\emptyset)+\sum_{J\subseteq(r,s],J\neq\emptyset}\nu_{r,s}(J)\\&=\nu(r,s)+\sum_{v=r+1}^{s}\sum_{J\subseteq(r,s],J\neq\emptyset,\min J=v}\nu_{r,s}(J)\\&=\nu(r,s)+\sum_{v=r+1}^{s}\nu'(r,v-1)\sum_{J\subseteq(v,s]}\nu_{v,s}(J)\\&=\nu(r,s)+\sum_{v=r+1}^{s}\nu'(r,v-1)\varepsilon(v,s),\end{aligned}$$

in view of (iii). $\square$

**Definition 5.7.** Let $J=\{j_1<j_2<\dots<j_f\}\subseteq(r,s]$. We set

$$\begin{aligned}\varepsilon(J)=\varepsilon_{r,s}(J)&=\varepsilon'(r,j_1-1)\varepsilon'(j_1,j_2-1)\dots\varepsilon'(j_{f-1},j_f-1)\varepsilon(j_f,s),\\\varepsilon'(J)=\varepsilon'_{r,s}(J)&=\varepsilon'(r,j_1-1)\varepsilon'(j_1,j_2-1)\dots\varepsilon'(j_{f-1},j_f-1)\varepsilon'(j_f,s),\end{aligned}$$

if $J\neq\emptyset$, and

$$\varepsilon(\emptyset)=\varepsilon_{r,s}(\emptyset)=\varepsilon(r,s),\quad \varepsilon'(\emptyset)=\varepsilon'_{r,s}(\emptyset)=\varepsilon'(r,s).$$

**Corollary 5.8.**

(i) $\nu(r,s)=\sum_{J\subseteq(r,s]}(-1)^{|J|}\varepsilon(J)$;
(ii) $\nu(r,s)=\varepsilon(r,s)-\sum_{v=r+1}^{s}\varepsilon'(r,v-1)\nu(v,s)$.

*Proof.* By Corollary 5.6(iv) we have

$$\nu(r,s)=\varepsilon(r,s)-\sum_{v=r+1}^{s}\nu'(r,v-1)\varepsilon(v,s).\tag{58}$$

We use induction on $s-r$ to prove (i). If $s=r$ then it follows from (58) that $\nu(r,r)=\varepsilon(r,r)=\varepsilon_{r,r}(\emptyset)$, as desired. Assume $s-r>0$, and (i) has been proved for all $r', s'$ with $s'-r'<s-r$. Then it follows from (58) that

$$\begin{aligned}\nu(r,s) &= \varepsilon_{r,s}(\emptyset) - \sum_{v=r+1}^{s} \sum_{J\subseteq(r,v-1]} (-1)^{|J|}\varepsilon'_{r,v-1}(J)\varepsilon(v,s)\\ &= \varepsilon_{r,s}(\emptyset) + \sum_{v=r+1}^{s} \sum_{J\subseteq(r,s],J\neq\emptyset,\max J=v} (-1)^{|J|}\varepsilon_{r,s}(J)\\ &= \sum_{J\subseteq(r,s]} (-1)^{|J|}\varepsilon_{r,s}(J).\end{aligned}$$

Thus, (i) is proved. To prove (ii) note that

$$\begin{aligned}\sum_{J\subseteq(r,s]} (-1)^{|J|}\varepsilon(J) &= \varepsilon_{r,s}(\emptyset) + \sum_{J\subseteq(r,s],J\neq\emptyset} (-1)^{|J|}\varepsilon_{r,s}(J)\\ &= \varepsilon(r,s) + \sum_{v=r+1}^{s} \sum_{J\subseteq(r,s],J\neq\emptyset,\min J=v} (-1)^{|J|}\varepsilon_{r,s}(J)\\ &= \varepsilon(r,s) - \sum_{v=r+1}^{s} \varepsilon'(r,v-1) \sum_{J\subseteq(v,s]} (-1)^{|J|}\varepsilon_{v,s}(J)\\ &= \varepsilon(r,s) - \sum_{v=r+1}^{s} \varepsilon'(r,v-1)\nu(v,s),\end{aligned}$$

in view of (i). □

If we fix positions $i$ between $r$ and $s$ with $\lambda_i \not\equiv 0$ then the recurrence formula from Proposition 5.5 can be improved:

**Corollary 5.9.** *Assume that*

$$\{i_1 < i_2 < \cdots < i_k\} = \{i \in [r,s] \mid \lambda_i \not\equiv 0\}.$$

*Then* $\varepsilon(r,s)=1$ *if* $k=1$, *and*

$$\begin{aligned}\varepsilon(r,s) =& \sigma(i_1,i_k) + (i_2-i_1)\varepsilon(i_2,i_k)\\ &+ \sum_{v=2}^{k-1} \Big((i_{v+1}-i_v-1)\sigma(i_1,i_v) + \sigma(i_1,i_{v-1})\Big)\varepsilon(i_{v+1},i_k)\end{aligned}$$

*if* $k \geq 2$.

*Proof.* By Lemma 2.3, $\varepsilon(r,s)=\varepsilon(i_1,i_k)$. If $k=1$ then $\varepsilon(i_1,i_k)=\varepsilon(i_1,i_1)=1$, and we are done. Let $k\geq 2$. Then by Proposition 5.5, we have

$$\begin{aligned}\varepsilon(i_1,i_k)=&\sigma(i_1,i_k)+\sum_{t=i_1+1}^{i_k}\sigma(i_1,t-2)\varepsilon(t,i_k)\\ =&\sigma(i_1,i_k)+\sum_{t=i_1+1}^{i_2}\sigma(i_1,t-2)\varepsilon(t,i_k)\\ &+\sum_{v=2}^{k-1}\left(\sigma(i_1,i_v-1)\varepsilon(i_v+1,i_k)+\sum_{t=i_v+2}^{i_{v+1}}\sigma(i_1,t-2)\varepsilon(t,i_k)\right).\end{aligned}$$

Let $t\in(i_1,i_2]$. By Lemma 2.3 we have $\varepsilon(t,i_k)=\varepsilon(i_2,i_k)$ since $\lambda_{i_1+1}\equiv\lambda_{i_1+2}\equiv\dots\equiv\lambda_{i_2-1}\equiv 0$. Moreover, $\sigma(i_1,t-2)=\sigma(i_1,i_1)=1$, in view of Definition 1.16 and Lemma 3.3. So

$$\sum_{t=i_1+1}^{i_2}\sigma(i_1,t-2)\varepsilon(t,i_k)=(i_2-i_1)\varepsilon(i_2,i_k).$$

Similarly we get

$$\sigma(i_1,i_v-1)=\sigma(i_1,i_{v-1}),\quad \varepsilon(i_v+1,i_k)=\varepsilon(i_{v+1},i_k),\quad v=2,3,\dots,k-1;$$

and

$$\sigma(i_1,t-2)=\sigma(i_1,i_v),\quad \varepsilon(t,i_k)=\varepsilon(i_{v+1},i_k),\quad t\in[i_v+2,i_{v+1}],\quad v=2,3,\dots,k-1.$$

So

$$\sum_{t=i_v+2}^{i_{v+1}}\sigma(i_1,t-2)\varepsilon(t,i_k)=(i_{v+1}-i_v-1)\sigma(i_1,i_v)\varepsilon(i_{v+1},i_k),$$

and the statement follows. □

*Remark.* Let $\mathcal{T}(r,s)=\mathcal{T}_\lambda(r,s)$ be the following $(s-r+1)$ by $(s-r+1)$ matrix

$$\begin{bmatrix}\nu'(r,r) & \nu'(r,r+1) & \cdots & \nu'(r,s-1) & \nu(r,s)\\ -1 & \nu'(r+1,r+1) & \cdots & \nu'(r+1,s-1) & \nu(r+1,s)\\ 0 & -1 & \cdots & \nu'(r+2,s-1) & \nu(r+2,s)\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & \nu'(s-1,s-1) & \nu(s-1,s)\\ 0 & 0 & \cdots & -1 & \nu(s,s)\end{bmatrix}.$$

Then Corollary 5.6(iii) can be reformulated as follows

$$\varepsilon(r,s)=\det\mathcal{T}(r,s). \tag{59}$$

Right hand sides of the expressions for $\varepsilon(r,s)$ from Corollary 5.6(i) and Corollary 5.9 can also be interpreted in terms of certain determinants (this is done in Theorem 1.18).

## 6. Quasi-latticed sequences and sets

**Definition 6.1.** Let $T = \{t_1, t_2, \ldots, t_m\}$ be a finite sequence with $t_i =$'+1','−1' or '0'. We call such a sequence *quasi-sign sequence*. (The empty sequence is quasi-sign by definition.)

**Definition 6.2.** Let $T = \{t_1, t_2, \ldots, t_m\}$ be a quasi-sign sequence. We call it *quasi-latticed* if and only if $\sum_{i=1}^{j} t_i \geq 0$ for all $j = 1, 2, \ldots, m$. (The empty sequence is quasi-latticed by definition.)

**Definition 6.3.** Let $T = \{t_1, t_2, \ldots, t_m\}$ be a quasi-sign sequence.

(i) For $1 \leq i \leq j \leq m$ we denote by $T(i, j)$ the subsequence $\{t_i, t_{i+1}, \ldots, t_j\}$.

(ii) Let $J$ be a subset in $[1, m]$. We call $J$ a *quasi-latticed subset* (in $[1, m]$ for $T$) if and only if the following two conditions hold

  (a) $J \subseteq \{i \mid t_i = -1 \text{ or } t_i = \pm\}$;

  (b) Let $[1, m] \setminus J = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_N, c_N]$ be the decomposition into the union of connected components (see 1.7). Then the sequences $T(b_i, c_i),\ i = 1, 2, \ldots, N$, are quasi-latticed.

(iii) We denote by $q(T)$ the number of quasi-latticed subsets in $[1, m]$ for $T$.

*Remark.* If $T = \emptyset$ we interpret $q(T) = 1$ (since the empty subset is the only quasi-latticed subset in $[1, 0] = \emptyset$).

Recall that throughout Part I we assume that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{Z}_+^l$.

**Definition 6.4.** We define a quasi-sign sequence $T_{r,s}(\lambda)$ as follows. Let

$$\{i_1 < i_2 < \cdots < i_m\}$$

be the set of all $i \in (r, s)$ with $\lambda_i > 0$ and either $\mathcal{B}_\lambda(r, i) \equiv 0$ or $\mathcal{C}_\lambda(r, i) \equiv 0$ (or both). Then

$$T_{r,s}(\lambda) = \{t_1, t_2, \ldots, t_m\}$$

where

$$t_j = \begin{cases} +1 & \text{if } \ \mathcal{B}_\lambda(r, i) \not\equiv 0,\ \mathcal{C}_\lambda(r, i) \equiv 0; \\ -1 & \text{if } \ \mathcal{B}_\lambda(r, i) \equiv 0,\ \mathcal{C}_\lambda(r, i) \not\equiv 0; \\ 0 & \text{if } \ \mathcal{B}_\lambda(r, i) \equiv 0,\ \mathcal{C}_\lambda(r, i) \equiv 0. \end{cases}$$

*Remark.* (a) If both $\mathcal{B}_\lambda(r, i) \equiv 0$ and $\mathcal{C}_\lambda(r, i) \equiv 0$ then we have $\lambda_i \equiv 0$.

(b) $T_{r,s}(\lambda)$ may be empty. For example $T_{r,r}(\lambda)$ is always empty.

**Definition 6.5.** We define the integer $q_{r,s}(\lambda)$ as follows. Put

$$q_{r,s}(\lambda) = 0$$

if at least one of the following conditions holds.

(a) $\lambda_r = 0$;

(b) $\lambda_s = 0$;

(c) $\mathcal{B}_\lambda(r, s) \not\equiv 0$.

Let $\lambda_r > 0,\ \lambda_s > 0$ and $\mathcal{B}_\lambda(r,s) \equiv 0$. Then we put

$$q_{r,s}(\lambda) = q(T_{r,s}(\lambda)),$$

see 6.4, 6.3(iii).

In what follows we denote

$$\begin{aligned} q_{r,s} &= q_{r,s}(\lambda), \\ q'_{r,s} &= q_{r,s}(\lambda(s)) \quad (\text{see } (5)). \end{aligned}$$

**Lemma 6.6.** *Let $T = \{t_1, t_2, \dots, t_m\}$ be a quasi-sign sequence, and put $T' = \{t_1, t_2, \dots, t_m, +1\}$. Then $q(T) = q(T')$.*

*Proof.* This follows immediately from Definition 6.3. □

**Lemma 6.7.** *Let $T = \{t_1, t_2, \dots, t_m\}$ be a quasi-sign sequence. For any $j \in [1, m]$ put*

$$T_j^- = \{t_1, t_2, \dots, t_{j-1}\}, \quad T_j^+ = \{t_{j+1}, t_{j+2}, \dots, t_m\}.$$

*Also put*

$$T_{\leq 0} = \{j \in [1,m] \mid t_j = -1 \text{ or } t_j = 0\}.$$

*Then*

$$q(T) = x + \sum\{q(T_j^-) \mid j \in T_{\leq 0},\ T_j^+ \text{ is quasi-latticed}\}$$

*where $x = 1$ if $T$ is quasi-latticed, and $x = 0$ otherwise.*

*Proof.* The empty subset in $[1, m]$ is quasi-latticed if and only if $T$ is quasi-latticed, i.e. if and only if $x = 1$.

Let $J$ be a non-empty quasi-latticed subset in $[1, m]$. Put $j = \max J$. Then $j \in T_{\leq 0}$. Moreover $T_j^+$ is quasi-latticed, and $J \cap [1, j-1]$ is quasi-latticed in $[1, j-1]$ for $T_j^-$.

On the other hand, if $j \in T_{\leq 0}$ is such that $T_j^+$ is quasi-latticed, and $J'$ is a quasi-latticed subset in $[1, j-1]$ for $T_j^-$ then $J \cup \{j\}$ is quasi-latticed in $[1, m]$ for $T$ and $\max J \cup \{j\} = j$.

This proves the lemma. □

**Lemma 6.8.** *Let $r < s$. Then $r$ is normal in $[r, s-1]$ if and only if $\lambda_r \not\equiv 0$ and $T_{r,s}(\lambda)$ is quasi-latticed.*

*Proof.* Let

$$\begin{aligned} &\{i_1, i_2, \dots, i_m\} = \{i \in (r,s) \mid \lambda_i > 0, \ \text{ and } \mathcal{B}(r,i) \equiv 0 \text{ or } \mathcal{C}(r,i) \equiv 0\}, \\ &T = T_{r,s}(\lambda) = \{t_1, t_2, \dots, t_m\}, \end{aligned}$$

see Definition 6.4. Put

$$J^+ \ (\text{resp.}, J^-, J^0) = \{j \in [1,m] \mid t_j = +1 \ (\text{resp.}, \ -1, 0)\}.$$

In view of 1.16 we may assume that $\lambda_r \not\equiv 0$.

Note that

$$I := \{i \in (r, s-1] \mid \lambda_i \not\equiv 0,\ \mathcal{B}(r,i) \equiv 0\} = \{i_j \mid j \in J^-\}.$$

Indeed, let $j \in J^-$. Then $\mathcal{B}(r, i_j) \equiv 0$ by definition (see 6.4). Moreover,

$$\lambda_{i_j} = \mathcal{B}(r, i_j) - \mathcal{C}(r, i_j) \equiv -\mathcal{C}(r, i_j) \not\equiv 0.$$

So $i_j \in I$. In the other direction, if $\mathcal{B}(r, i) \equiv 0$ then $i = i_j$ for $j \in J^- \cup J^{\pm}$. However, $\lambda_i \not\equiv 0$ implies $\mathcal{C}(r, i) \not\equiv 0$ so $j \in J^-$.

Now, by virtue of Lemma 3.3, $r$ is normal in $[r, s-1]$ if and only if for all $i \in I$ there exist $f(i) \in (r, i)$, all distinct, such that $\lambda_{f(i)} \not\equiv 0$ and $\mathcal{B}(f(i), i) \equiv 0$. For such $f(i)$ we have

$$\mathcal{C}(r, f(i)) = \mathcal{B}(r, i) - \mathcal{B}(f(i), i) \equiv 0.$$

So $f(i) = i_{d(i)}$ for some $d(i) \in J^+ \cup J^{\pm}$. Moreover, $\lambda_{f(i)} \not\equiv 0$ implies

$$\mathcal{B}(r, f(i)) = \mathcal{C}(r, f(i)) + \lambda_{f(i)} \equiv \lambda_{f(i)} \not\equiv 0.$$

So $d(i) \in J^+$. Thus if $r$ is normal in $[r, s-1]$ then for all $j \in J^-$ there exist $d(j) \in J^+$, all distinct, with $1 \leq d(j) < j$. This is equivalent to the fact that $T$ is quasi-latticed. The argument is easily reversed. $\square$

To describe some decomposition numbers below we shall need the following recurrence formula for $q_{r,s}$. Recall the constants $\bar{\sigma}(i,j)$ defined in (31).

**Proposition 6.9.** *Assume* $r < s$, $\lambda_r > 0$, $\lambda_s > 0$, $\mathcal{B}_\lambda(r, s) \equiv 0$. *Then*

$$q_{r,s} = q_{r,s-1} + \sigma(r, s-1) + \sum_{i=r}^{s-2} q_{r,i}\bar{\sigma}(i+1, s-1).$$

*Moreover, if* $\lambda_s \equiv 0$ *then*

$$q_{r,s} = q'_{r,s-1}.$$

*Proof.* It follows from Definition 6.5 that $q_{r,s} = q(T_{r,s}(\lambda))$.

Assume first that $\lambda_s \equiv 0$. Then

$$\begin{aligned}
\mathcal{B}_{\lambda(s-1)}(r, s-1) &= \sum_{u=r}^{s-2} \lambda_u + (\lambda_{s-1} + 1) + (s-1) - r \\
&= \sum_{u=r}^{s-1} \lambda_u + s - r \equiv \sum_{u=r}^{s} \lambda_u + s - r \\
&= \mathcal{B}_\lambda(r, s) \equiv 0.
\end{aligned}$$

So $q'_{r,s-1} = q_{r,s-1}\Big(T_{r,s-1}(\lambda(s-1))\Big)$, see Definition 6.5. Let

$$T_{r,s-1}(\lambda(s-1)) = \{u_1, u_2, \ldots, u_m\}.$$

Note that

$$\begin{aligned}
\mathcal{B}_\lambda(r, s-1) &= \mathcal{B}_\lambda(r, s) - (\lambda_s + 1) \equiv -1 \not\equiv 0, \\
\mathcal{C}_\lambda(r, s-1) &= \mathcal{B}_\lambda(r, s-1) - \lambda_{s-1}.
\end{aligned}$$

So $\mathcal{C}_\lambda(r,s-1)\equiv 0$ implies $\lambda_{s-1}\not\equiv 0$. Therefore either $T_{r,s}(\lambda)=T_{r,s-1}(\lambda(s-1))$ or $T_{r,s}(\lambda)=\{u_1,u_2,\dots,u_m,+1\}$. By Lemma 6.6 we have $q(T_{r,s}(\lambda))=q\Big(T_{r,s-1}(\lambda(s-1))\Big)$. This proves $q_{r,s}=q'_{r,s-1}$.

Now, let us consider the general case (i.e. $\lambda_s>0$ is arbitrary). Note that

$$\mathcal{B}_\lambda(r,j)+\mathcal{B}_{\lambda(j+1)}(j+1,i)=\mathcal{B}_\lambda(r,i),$$
$$\mathcal{B}_\lambda(r,j)+\mathcal{C}_{\lambda(j+1)}(j+1,i)=\mathcal{C}_\lambda(r,i)$$

for any $i,j$ such that $r\le j<i\le s$. So we get

$$\begin{aligned}&\{i\in[j+1,s-1]\mid \mathcal{B}_\lambda(r,i)\equiv 0\}\\ &=\{i\in[j+1,s-1]\mid \mathcal{B}_{\lambda(j+1)}(j+1,i)\equiv 0\}\\ &\text{for any } j\in[r,s) \text{ with } \mathcal{B}_\lambda(r,j)\equiv 0.\end{aligned}\tag{60}$$

$$\begin{aligned}&\{i\in[j+1,s-1]\mid \mathcal{C}_\lambda(r,i)\equiv 0\}\\ &=\{i\in[j+1,s-1]\mid \mathcal{C}_{\lambda(j+1)}(j+1,i)\equiv 0\}\\ &\text{for any } j\in[r,s) \text{ with } \mathcal{B}_\lambda(r,j)\equiv 0.\end{aligned}\tag{61}$$

We put

$$\{i_1<i_2<\dots<i_m\}=\{i\in(r,s)\mid \lambda_i>0,\ \text{and } \mathcal{B}_\lambda(r,i)\equiv 0 \text{ or } \mathcal{C}_\lambda(r,i)\equiv 0\},$$

and

$$T=T_{r,s}(\lambda)=\{t_1,t_2,\dots,t_m\}.$$

Then by Lemma 6.7 we have

$$q_{r,s}=q(T)=x+\sum\{q(T_j^-)\mid j\in T_{\le 0},\ T_j^+ \text{ is quasi-latticed}\}\tag{62}$$

where $x=1$ if $T$ is quasi-latticed, and $x=0$ otherwise.

We claim that

$$x=\sigma(r,s-1)+q_{r,r}\bar\sigma(r+1,s-1).\tag{63}$$

Indeed, assume first $\lambda_r\not\equiv 0$. Then $q_{r,r}=0$. Moreover, $T$ is quasi-latticed if and only if $r$ is normal in $[r,s-1]$ in view of Lemma 6.8. So $x=\sigma(r,s-1)$ (cf. Definition 1.16).

Now assume $\lambda_r\equiv 0$. Then $\sigma(r,s-1)=0$, $q_{r,r}=1$ (since $\lambda_r>0$). We have to consider two subcases: $s=r+1$ and $s>r+1$.

If $s=r+1$ then $\bar\sigma(r+1,s-1)=1$ by Definition 1.16, so the right hand side of (63) is equal to 1. However, we also have $x=1$ since the empty sequence is quasi-latticed by Definition 6.2.

Now consider the second subcase: $s>r+1$. We again need to distinguish between two possibilities: $\lambda_{r+1}+1\equiv 0$ and $\lambda_{r+1}+1\not\equiv 0$.

If $\lambda_{r+1}+1\equiv 0$ then $\lambda_{r+1}\not\equiv 0$ and $\mathcal{B}_\lambda(r,r+1)=\lambda_r+\lambda_{r+1}+1\equiv 0$ since we have assumed $\lambda_r\equiv 0$. So $i_1=r+1$, $t_1=-1$. Therefore $T$ is not quasi-latticed, i.e. $x = 0$. On the other hand $\lambda_{r+1} + 1 \equiv 0$ implies that $r+1$ is not normal in $[r+1, s-1]$ for $\lambda(r+1)$, i.e. $\bar\sigma(r+1, s-1) = 0$.

Finally, let $\lambda_{r+1} + 1 \not\equiv 0$. Since $\mathcal{B}_\lambda(r, r) = \lambda_r \equiv 0$, equalities (60) and (61) imply

$$\{i \in (r+1, s) \mid \mathcal{B}_{\lambda(r+1)}(r+1, i) \equiv 0\} = \{i \in (r+1, s) \mid \mathcal{B}_\lambda(r, i) \equiv 0\},$$
$$\{i \in (r+1, s) \mid \mathcal{C}_{\lambda(r+1)}(r+1, i) \equiv 0\} = \{i \in (r+1, s) \mid \mathcal{C}_\lambda(r, i) \equiv 0\}.$$

Moreover,

$$\mathcal{B}_\lambda(r, r+1) = \lambda_r + \lambda_{r+1} + 1 \not\equiv 0 \quad \text{and} \quad \mathcal{C}_\lambda(r, r+1) = \lambda_r + 1 \equiv 1 \not\equiv 0.$$

So $T_{r,s}(\lambda) = T_{r+1,s}(\lambda(r+1))$. But Lemma 6.8 implies that $T_{r+1,s}(\lambda(r+1))$ is quasi-latticed if and only if $r+1$ is normal in $[r+1, s-1]$ for $\lambda(r+1)$, i.e. if and only if $\bar\sigma(r+1, s-1) = 1$. On the other hand, $x = 1$ if and only if $T_{r,s}(\lambda)$ is quasi-latticed. Since $T_{r,s}(\lambda) = T_{r+1,s}(\lambda(r+1))$ this proves the equality $x = \bar\sigma(r+1, s-1)$.

Thus (63) is completely proved.

We next claim that

$$\sum\{q(T_j^-) \mid j \in T_{\le 0},\ T_j^+ \text{ is quasi-latticed}\} = \sum_{j \in T_{\le 0}} q(T_j^-)\bar\sigma(i_j + 1, s-1). \tag{64}$$

Let $j \in T_{\le 0}$. If $i_j = s-1$ then $j = m$, $T_j^+ = \emptyset$ is quasi-latticed. Moreover, $\bar\sigma(i_j + 1, s-1) = \bar\sigma(s, s-1) = 1$ by definition. So $q(T_j^-)$ occurs in both sides of (64).

Assume $i_j < s-1$. Since $j \in T_{\le 0}$ we have $\mathcal{B}(r, i_j) \equiv 0$. It follows from (60), (61) that

$$\begin{aligned} &\{i \in (i_j+1, s-1] \mid \mathcal{B}_{\lambda(i_j+1)}(i_j+1, i) \equiv 0\} \\ &= \{i \in (i_j+1, s-1] \mid \mathcal{B}_\lambda(r, i) \equiv 0\}, \end{aligned} \tag{65}$$

$$\begin{aligned} &\{i \in (i_j+1, s-1] \mid \mathcal{C}_{\lambda(i_j+1)}(i_j+1, i) \equiv 0\} \\ &= \{i \in (i_j+1, s-1] \mid \mathcal{C}_\lambda(r, i) \equiv 0\}, \end{aligned} \tag{66}$$

Note that $\mathcal{B}_\lambda(r, i_j) \equiv 0$ also implies

$$\mathcal{C}_\lambda(r, i_j + 1) = \mathcal{B}_\lambda(r, i_j) + 1 \equiv 1 \not\equiv 0, \tag{67}$$

$$\mathcal{B}_\lambda(r, i_j + 1) = \mathcal{B}_\lambda(r, i_j) + (\lambda_{i_j+1} + 1) \equiv \lambda_{i_j+1} + 1. \tag{68}$$

If $\lambda_{i_j+1} + 1 \equiv 0$ then $\lambda_{i_j+1} \not\equiv 0$, and $\mathcal{B}_\lambda(r, i_j+1) \equiv 0$ by (68). Since $i_j + 1 < s$, this implies $j < m$, $i_{j+1} = i_j + 1$ and $t_{j+1} = -1$. So $T_j^+$ is not quasi-latticed, i.e. $q(T_j^-)$ does not occur in the left hand side of (64). On the other hand, $\lambda_{i_j+1} + 1 \equiv 0$ implies $\bar\sigma(i_j + 1, s-1) = 0$, so $q(T_j^-)$ also does not occur in the right hand side of (64).

Let $\lambda_{i_j} + 1 \not\equiv 0$. Then (68) implies $\mathcal{B}_\lambda(r, i_j + 1) \not\equiv 0$, which together with (67) and (65),(66) implies

$$T_j^+ = T_{i_j+1,s}(\lambda(i_j+1)). \tag{69}$$

By Lemma 6.8, $T_{i_j+1,s}(\lambda(i_j+1))$ is quasi-latticed if and only if $i_j+1$ is normal in $[i_j+1, s-1]$ for $\lambda(i_j+1)$, i.e. if and only if $\bar\sigma(i_j+1, s-1) = 1$. So it follows from (69) that $q(T_j^-)$ occurs in the left hand side of (64) if and only if it occurs in its right hand side. Thus (64) is completely proved.

Further, note that $T_j^- = T_{r,i_j}(\lambda)$ for any $j \in [1, m]$. So for $j \in T_{\leq 0}$ we have $q(T_j^-) = q_{r,i_j}$. Thus it follows from (62), (63) and (64) that

$$q_{r,s} = \sigma(r, s-1) + q_{r,r}\bar\sigma(r+1, s-1) + \sum_{j\in T_{\leq 0}} q_{r,i_j}\bar\sigma(i_j+1, s-1). \tag{70}$$

Assume $i \in (r, s)$. Then $q_{r,i} = 0$ unless $\mathcal{B}(r, i) \equiv 0$, $\lambda_i > 0$, i.e. $q_{r,i} = 0$ unless $i = i_j$ for some $j \in T_{\leq 0}$. So we may rewrite (70) as

$$q_{r,s} = \sigma(r, s-1) + q_{r,r}\bar\sigma(r+1, s-1) + \sum_{i=r+1}^{s-1} q_{r,i}\bar\sigma(i+1, s-1).$$

Since $\bar\sigma(s, s-1) = 1$ this is equivalent to

$$q_{r,s} = \sigma(r, s-1) + q_{r,s-1} + \sum_{i=r}^{s-2} q_{r,i}\bar\sigma(i+1, s-1),$$

as desired. $\square$

## Part II. Main Results

Throughout Part II:

$l$ is a fixed positive integer;

$r$ and $s$ are always assumed to be integers with $1 \leq r \leq s \leq l$;

$K$ is an algebraically closed field of characteristic $p > 0$;

$\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ is a base of the root system $A_l$;

$\{\omega_1, \omega_2, \ldots, \omega_l\}$ are the corresponding fundamental weights;

$P(A_l) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_l$ is the weight lattice;

$P_+(A_l) = \mathbb{Z}_+\omega_1 + \mathbb{Z}_+\omega_2 + \cdots + \mathbb{Z}_+\omega_l$ is the set of dominant weights;

$\alpha(r, s) = \alpha_r + \alpha_{r+1} + \cdots + \alpha_s \in A_l$, $1 \leq r \leq s \leq l$. (Recall that any root in $A_l$ has a form $\pm\alpha(r, s)$);

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_l\omega_l \in P_+(A_l)$.

### 7. Preliminaries from representation theory

We consider the standard Chevalley basis

$$\{E_{i,j}\ (1 \leq i \neq j \leq l+1),\ E_{i,i} - E_{i+1,i+1}\ (1 \leq i \leq l)\}$$

of the Lie algebra $\mathfrak{sl}_{l+1}(\mathbb{C})$ (see Remarks (a), (c) after Theorem 1 in [24]). We define a $\mathbb{Z}$-algebra $\mathcal{U}_\mathbb{Z}$ (resp., $\mathcal{U}_\mathbb{Z}{}^+$, $\mathcal{U}_\mathbb{Z}{}^-$) as the $\mathbb{Z}$-subalgebra of the universal enveloping $\mathcal{U}(\mathfrak{sl}_{l+1}(\mathbb{C}))$ generated by all $(E_{i,j})^m/m!$, $1 \leq i \neq j \leq l+1$ (resp., $1 \leq i < j \leq l+1$, $1 \leq j < i \leq l+1$). Put $\mathcal{U}_K = \mathcal{U}_\mathbb{Z} \otimes K$.

We define

$$\begin{aligned} X_{r,s}^{(m)} &= ((E_{r,s+1})^m/m!) \otimes 1 \in \mathcal{U}_K, \quad X_{r,s} = X_{r,s}^{(1)};\\ Y_{r,s}^{(m)} &= ((E_{s+1,r})^m/m!) \otimes 1 \in \mathcal{U}_K, \quad Y_{r,s} = Y_{r,s}^{(1)};\\ H_{r,s} &= (E_{r,r} - E_{s+1,s+1}) \otimes 1 \in \mathcal{U}_K. \end{aligned}$$

(Recall that we always assume $1 \le r \le s \le l$).

For a weight $\mu \in P(A_l)$ we define $\mathcal{U}_\mu^+$ and $\mathcal{U}_{-\mu}^-$ to be generated (as vecor spaces) by the products

$$\left\{ X_{i_1,j_1}^{(m_1)} X_{i_2,j_2}^{(m_2)} \dots X_{i_N,j_N}^{(m_N)} \,\middle|\, \sum_{t=1}^{N} m_t \alpha(i_t, j_t) = \mu \right\}$$

and

$$\left\{ Y_{i_1,j_1}^{(m_1)} Y_{i_2,j_2}^{(m_2)} \dots Y_{i_N,j_N}^{(m_N)} \,\middle|\, \sum_{t=1}^{N} m_t \alpha(i_t, j_t) = \mu \right\},$$

respectively.

Let $A_l(K)$ be the universal Chevalley group defined using the fixed Chevalley basis and considered as an algebraic group [24, Sections 3,5]. All modules over $A_l(K)$ are assumed to be rational and finite-dimensional $K$-vector spaces. Then we may not to distinguish between $A_l(K)$- and $\mathcal{U}_K$-modules, cf. [11].

If $M$ is an $A_l(K)$-module and $\mu \in P(A_l)$ we denote by $M_\mu$ the $\mu$-weight space of $M$.

Let $V_{\mathbb{C}}(\lambda)$ be the complex irreducible $\mathfrak{sl}_{l+1}(\mathbb{C})$-module with highest weight $\lambda$ (recall that throughout Part II we assume $\lambda \in P_+(A_l)$). We denote by $e_{\mathbb{C}}$ a highest weight vector of $V_{\mathbb{C}}(\lambda)$. Set

$$\begin{aligned} &V_{\mathbb{Z}}(\lambda) = \mathcal{U}_{\mathbb{Z}} e_{\mathbb{C}}, \quad V(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes K,\\ &e_+ = e_{\mathbb{C}} \otimes 1 \in V(\lambda). \end{aligned}$$

Then $V(\lambda)$ is a $\mathcal{U}_K$-module called Weyl module with highest weight $\lambda$. This module has a unique maximal proper submodule $\Theta(\lambda)$, and

$$L(\lambda) = V(\lambda)/\Theta(\lambda)$$

is the irreducible module with highest weight $\lambda$. We denote by

$$v_+ = e_+ + \Theta(\lambda) \in V(\lambda)/\Theta(\lambda) = L(\lambda)$$

a highest weight vector of $L(\lambda)$.

The next result is well known (It follows easily from [24, Theorem 2] and the fact that $L(\lambda) = \mathcal{U}_K L(\lambda)$.)

**Lemma 7.1.** $L(\lambda)_{\lambda-\mu} = \mathcal{U}_{-\mu}^- v_+$.

We need to recall the notion of the first level introduced by Seitz.

**Definition 7.2.** *The first level* $L(\lambda)_1$ of an irreducible module $L(\lambda)$ is defined as

$$L(\lambda)_1 = \bigoplus \{L(\lambda)_\mu | \mu = \lambda - \alpha_l - \sum_{i=1}^{l-1} m_i \alpha_i \text{ for some } m_i \in \mathbb{Z}\}.$$

Let $A_{l-1}(K)$ be a subgroup of $A_l(K)$ generated by the root subgroups which correspond to all roots of the form $\pm\alpha(i,j)$, $1 \leq i \leq j < l$. Then $L(\lambda)_1$ is a direct summand of the restriction $L(\lambda) \downarrow_{A_{l-1}(K)}$. In particular, we can consider $L(\lambda)_1$ as an $A_{l-1}(K)$-module (see [12, 13] for more details). This module will be denoted by $L(\lambda)_1 \mid A_{l-1}(K)$.

If

$$\mu = (\mu_1, \mu_2, \dots, \mu_l) \in P(A_l)$$

we denote

$$\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_{l-1}) \in P(A_{l-1}).$$

Being considered as a character of the standard Cartan subgroups of $A_{l-1}(K) \subset A_l(K)$, $\bar{\mu}$ is just a restriction of $\mu$ (see [13] for more details).

**Lemma 7.3.** *Let* $\mu, \nu \in P(A_l)$, $\mu = \lambda - \alpha_l - \sum_{i=1}^{l-1} n_i \alpha_i$, $\nu = \lambda - \alpha_l - \sum_{i=1}^{l-1} m_i \alpha_i$. *If* $\bar{\mu} = \bar{\nu}$ *then* $\mu = \nu$.

*Proof.* Let $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, $\nu = (\nu_1, \nu_2, \dots, \nu_l)$. Then by [2, Table I] we have

$$\begin{aligned}
\mu_1 &= \lambda_1 - 2n_1 + n_2;\\
\mu_i &= \lambda_i - 2n_i + n_{i-1} + n_{i+1},\ i = 2, 3, \dots, l-2;\\
\mu_{l-1} &= \lambda_{l-1} - 2n_{l-1} + n_{l-2} + 1;\\
\mu_l &= \lambda_l - 2 + n_{l-1};
\end{aligned}$$

and

$$\begin{aligned}
\nu_1 &= \lambda_1 - 2m_1 + m_2;\\
\nu_i &= \lambda_i - 2m_i + n_{i-1} + n_{i+1},\ i = 2, 3, \dots, l-2;\\
\nu_{l-1} &= \lambda_{l-1} - 2m_{l-1} + m_{l-2} + 1;\\
\nu_l &= \lambda_l - 2 + m_{l-1}.
\end{aligned}$$

By assumption, $\mu_i = \nu_i$ for $i = 1, 2, \dots, l-1$. So

$$\begin{aligned}
&-2n_1 + n_2 = -2m_1 + m_2;\\
&-2n_i + n_{i-1} + n_{i+1} = -2m_i + m_{i-1} + m_{i+1},\ i = 2, 3, \dots, l-2;\\
&-2n_{l-1} + n_{l-2} = -2m_{l-1} + m_{l-2}.
\end{aligned}$$

Therefore $m_i = n_i$ for $i = 1, 2, \dots, l-1$. $\square$

**Lemma 7.4.** *Let* $\mu = \lambda - \sum_{i=1}^{l} n_i \alpha_i$. *Assume that* $n_l = 1$ *and* $n_i = 0$ *or* $1$ *for* $i = 1, 2, \dots, l-1$. *If* $L(\bar{\mu})$ *occurs as a composition factor of* $L(\lambda)_1 \mid A_{l-1}(K)$ *then* $\mu = \lambda - \alpha(t, l)$ *for some* $t \in [1, l]$.

*Proof.* Denote $L(\lambda)_1 \mid A_{l-1}(K)$ by $M$, and let

$$0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(N)} = M$$

be a composition series. Assume $M^{(j)}/M^{(j-1)} \cong L(\bar{\mu})$. We choose $v \in M^{(j)}$ such that $v + M^{(j-1)} \in M^{(j)}/M^{(j-1)}$ is a highest weight vector of $M^{(j)}/M^{(j-1)} \cong L(\bar{\mu})$.

Without loss of generality we may assume that $v \in L(\lambda)_\mu$. Indeed we may clearly assume that $v \in M_{\bar{\mu}}$. So

$$v = v_\nu + v_\kappa + \dots$$

where $v_\nu \in L(\lambda)_\nu$, $v_\kappa \in L(\lambda)_\kappa$, $\dots$, $\bar{\mu} = \bar{\nu} = \bar{\kappa} = \dots$, and $\nu, \kappa, \dots$ are weights of the first level. In view of Lemma 7.3, we have $\mu = \nu = \kappa = \dots$.

Now, let $t = \min\{i \mid 1 \le i \le l,\ n_i \neq 0\}$. Then $t$ is well defined since $n_l = 1$. We have to prove that $n_i = 1$ for $i = t, t+1, \dots, l$. If not, let $u \ge t$ be the maximal such that $n_i = 1$ for all $i \in [t, u]$. Then $u + 1 < l$ and $n_{u+1} = 0$.

Set $\eta = \sum_{i=1}^{l} n_i\alpha_i$ and $\delta = \sum_{i=u+2}^{l} n_i\alpha_i$. Then $\eta = \alpha(t, u) + \delta$.

Since $v \in L(\lambda)_\mu$, Lemma 7.1 implies that there exists $y \in \mathcal{U}^-_{-\eta}$ with $v = yv_+$. Note that $y = y_1y_2$ where $y_1 \in \mathcal{U}^-_{-\alpha(t,u)}$, $y_2 \in \mathcal{U}^-_{-\delta}$, and $y_1y_2 = y_2y_1$.

Since $v \neq 0$ there exists $x \in \mathcal{U}^+_\eta$ with $xv = v_+$. Again, $x = x_1x_2$ where $x_1 \in \mathcal{U}^+_{\alpha(t,u)}$, $x_2 \in \mathcal{U}^+_\delta$, and $x_1x_2 = x_2x_1$.

Clearly $x_1y_2 = y_2x_1$.

The equation $xv = v_+$ yields $x_1x_2y_1y_2v_+ = v_+$ which is equivalent to the equation $x_2y_2x_1y_1v_+ = v_+$. So $x_1y_1v_+ \neq 0$, i.e. $x_1y_1v_+ = cv_+$ for some $c \neq 0$. Since $v + M^{(j-1)}$ is a primitive vector in $M^{(j)}/M^{(j-1)}$ we have $x_1v \in M^{(j-1)}$. Since $M^{(j-1)}$ is an $A_{l-1}(K)$-submodule of $L(\lambda)_1 \mid A_{l-1}(K)$ we have $y_1x_1v \in M^{(j-1)}$.

On the other hand,

$$y_1x_1v = y_1x_1y_1y_2v_+ = y_1y_2x_1y_1v_+ = y_1y_2cv_+ = cv,$$

with $c \neq 0$. This gives a contradiction. □

**Definition 7.5.** Let $J = \{j_1 < j_2 < \cdots < j_f\} \subseteq (r, s]$. We define an element $Y_{r,s}(J) \in \mathcal{U}^-_{-\alpha(r,s)}$ by putting

$$Y_{r,s}(J) = Y_{r,j_1-1}Y_{j_1,j_2-1}\dots Y_{j_f,s}.$$

($Y_{r,s}(\emptyset)$ is interpreted as $Y_{r,s}$).

**Lemma 7.6.** *$\{Y_{r,s}(J) \mid J \subseteq (r, s]\}$ is a basis of $\mathcal{U}^-_{-\alpha(r,s)}$.*

*Proof.* This follows from [24, Theorem 2]. □

Recall the integers $\mathcal{B}(r, s), \mathcal{C}(r, s)$ defined in 1.11. In [14, 2.3] we have also defined integers $\zeta_{r,s}(M, U)$ for any two subsets $M, U \subseteq (r, s]$.

In what follows we use the (same) notation $\mathcal{B}(r, s), \mathcal{C}(r, s), \zeta_{r,s}(M, U)$, etc. for the elements $\mathcal{B}(r, s) \otimes 1, \mathcal{C}(r, s) \otimes 1, \zeta_{r,s}(M, U) \otimes 1$, etc., of $\mathbb{Z} \otimes K \cong K$.

For any $M \subseteq (r, s]$ put

$$T_{r,s}(M) = \sum_{U \subseteq (r,s]} \zeta_{r,s}(M, U)Y_{r,s}(U) \in \mathcal{U}^-_{-\alpha(r,s)}.$$

The elements $T_{r,s}(M)$ (more precisely their $\mathbb{Z}$-forms) were introduced in [14, 2.5,2.6] as some iterations of Carter's lowering operators. We formulate some important properties of $T_{r,s}(M)$ for future reference. All of them follow immediately from the similar properties of the corresponding $\mathbb{Z}$-forms which were proved in [14].

**Lemma 7.7.** [14, 2.17]

(i) *Let $J \subseteq (r,s]$. Set $\bar{J} = (r,s] \setminus J$. Then*

$$T_{r,s}(J) = Y_{r,s}(\bar{J}) + \sum_{I \subseteq (r,s], I < \bar{J}} \mathrm{const}_I \, Y_{r,s}(I)$$

*(recall the order '$<$' defined in* 2.5*).*

(ii) $\{T_{r,s}(J) \mid J \subseteq (r,s]\}$ *is a basis of* $\mathcal{U}^-_{-\alpha(r,s)}$.

**Lemma 7.8.** [14, 2.14] *Let $r < i \leq s$, $E \subseteq (r, i-1]$, $F \subseteq (i,s]$, $e = \max E$ if $E \neq \emptyset$. Then*

$$T_{r,i-1}(E)T_{i,s}(F) = T_{r,s}(E \cup F) - T_{r,s}((E \setminus \{e\}) \cup \{i\} \cup F) - \mathcal{C}(e,i)T_{r,s}(E \cup \{i\} \cup F),$$

*if $E \neq \emptyset$, and*

$$T_{r,i-1}(\emptyset)T_{i,s}(F) = T_{r,s}(F) - \mathcal{C}(r,i)T_{r,s}(\{i\} \cup F).$$

Recall that $\lambda \in P_+(A_l)$ is a (fixed) dominant weight, and $v_+$ is a highest weight vector vector of $L(\lambda)$.

**Lemma 7.9.** [14, 3.2] *We have $T_{r,s}(J)v_+ = 0$ if and only if there exists*

$$u \in (r, s+1] \setminus J$$

*and a weakly increasing bijection*

$$d : \{r\} \cup (J \cap (r,u)) \to [r,u)$$

*such that $\mathcal{B}(k, d(k)) \equiv 0$ for all $k \in \{r\} \cup (J \cap (r,u))$.*

For any $J \subseteq (r,s]$ there is a constant $\xi_{r,s}(J) \in K$ such that

$$X_{r,r}X_{r+1,r+1}\dots X_{s,s}T_{r,s}(J)e_+ = \xi_{r,s}(J)e_+$$

(recall that $e_+$ is a highest weight vector of $V(\lambda)$). We refer the reader to [14, 2.10,2.9] for details (there is a misprint in [14, Definition 2.9] though: we must put $\xi_{r,s}(\emptyset) = \prod_{t=r}^s \mathcal{B}(r,t)$).

**Lemma 7.10.** [14, 2.11] *Let $J \subseteq (r,s]$ be a regular subset (see Definition* 5.1*), i.e. there exist $d(j)$ for $j \in J$, all distinct, with $j \leq d(j)$, $\mathcal{B}(j, d(j)) \equiv 0$. Put $D = \{d(j) \mid j \in J\}$. Then*

$$\xi_{r,s}(J) = \prod_{t \in [r,s] \setminus D} \mathcal{B}(r,t).$$

In [14, 2.18], we also introduced certain elements $Q_{r,s}(J) \in \mathcal{U}^+_{\alpha(r,s)}$, $J \subseteq (r,s]$. We define constants $h_{r,s}(I,J) \in K$ by

$$Q_{r,s}(I)T_{r,s}(J)e_+ = h_{r,s}(I,J)e_+; \quad I, J \subseteq (r,s].$$

(see Definition 2.22 in [14])

Recall the partial order '$\ll$' from Definition 2.7(i).

**Lemma 7.11.** [14, 2.25] *Suppose that $h_{r,s}(I,J) \neq 0$. Then $I \ll J$.*

**Lemma 7.12.** [14, 2.28] *Let $I \ll J$ be as in Definition 2.7. Then*

$$h_{r,s}(I,J) = \pm \xi_{r,b_1-2}(\Delta_0(I,J))\xi_{c_H,s}(\Delta_H(I,J)) \prod_{t=1}^{H-1} \xi_{c_t,b_{t+1}-2}(\Delta_t(I,J)).$$

**Lemma 7.13.** [14, 2.29] *Let $I \ll J$ be as in Definition 2.7 and assume that $I$ is linked with $J$ (see 2.7(iii)). Then*

$$\begin{aligned} h_{r,s}(I,J) = & \Big[ \prod_{d_0 \in [r,b_1-2] \setminus \Omega_0(I,J)} \mathcal{B}(r,d_0) \Big] \\ & \times \Big[ \prod_{t=1}^{H-1} \prod_{d_t \in [c_t,b_{t+1}-2] \setminus \Omega_t(I,J)} \mathcal{B}(c_t,d_t) \Big] \Big[ \prod_{d_H \in [c_H,s] \setminus \Omega_H(I,J)} \mathcal{B}(c_H,d_H) \Big]. \end{aligned}$$

**Definition 7.14.** [14, 4.4] Assume that $r < s$ and there exists $i \in [r,s]$ such that $\lambda_i \not\equiv 0$. Then we say that $Y_{r,s}v_+$ *can be expressed through non-trivial subdivisions* if and only if there exists (in $L(\lambda)$) an expression of the form.

$$Y_{r,s}v_+ = \sum_{J \subseteq (r,s], J \neq \emptyset} c_J Y_{r,s}(J)v_+, \quad c_J \in K.$$

Recall the function $\sigma(r,s)$ defined in 1.16.

**Lemma 7.15.** [14, 4.6,4.2,4.3] *Assume that $r < s$ and there exists $i \in [r,s]$ such that $\lambda_i \not\equiv 0$. Then $Y_{r,s}v_+$ can be expressed through non-trivial subdivisions if and only if $\sigma(r,s) = 0$.*

## 8. Dimension and bases of the weight space $L(\lambda)_{\lambda-\alpha}$

The purpose of this section is to describe the weight space $L(\lambda)_{\lambda-\alpha}$ for any positive root $\alpha = \alpha_r + \alpha_{r+1} + \cdots + \alpha_s$. We find three distinct bases in this space and obtain a combinatorial description of its dimension. We also deduce a recurrence formula for $\dim L(\lambda)_{\lambda-\alpha}$ and investigate the question when $\dim L(\lambda)_{\lambda-\alpha}$ takes its minimal value.

**Lemma 8.1.** *A subset $J \subseteq (r,s]$ is non-essential if and only if there exists $i \in [r,s] \setminus J$ such that $T_{i,s}(J \cap (i,s])v_+ = 0$.*

*Proof.* Assume that we found $i \in [r,s] \setminus J$ such that $T_{i,s}(J \cap (i,s])v_+ = 0$. By Lemma 7.9, there exist $u \in (i, s+1] \setminus (J \cap (i,s])$ and integers $d$, $d(j)$ for $j \in (J \cap (i,s]) \cap (i,u) = J \cap (i,u)$, all distinct, such that

$$i \leq d < u, \quad \mathcal{B}(i,d) \equiv 0$$

and

$$j \leq d(j) < u, \quad \mathcal{B}(j, d(j)) \equiv 0 \quad \text{for} \quad j \in J \cap (i,u).$$

Now we may take

$$t = i, \quad d(t) = d, \quad u = u, \quad d(j) = d(j)$$

in Definition 1.13. Thus $J$ is not essential. The argument is easily reversed. $\square$

Set

$$\{T_{r,s}(J)v_+ \mid J \text{ is essential in } (r,s]\}. \tag{71}$$

We want to prove that (71) is a basis of the weight space $L(\lambda)_{\lambda - \alpha(r,s)}$. Corollary 8.3 is the first step in achieving this.

**Lemma 8.2.** *Assume $L(\lambda)_{\lambda-\alpha(r,s)} \neq 0$ and let $M \subseteq (r,s]$ be non-essential. Then $T_{r,s}(M)v_+$ can be expressed as a linear combination of $T_{r,s}(J)v_+$ with $J > M$ and $J$ essential.*

*Proof.* Note that $(r,s]$ is essential in $(r,s]$. Indeed, otherwise it follows from Definition 1.13 that $\lambda_r \equiv \lambda_{r+1} \equiv \cdots \equiv \lambda_s \equiv 0$, whence $L(\lambda)_{\lambda-\alpha(r,s)} = 0$. Thus there is at least one essential subset in $(r,s]$.

Assume the statement of the lemma is false. Let $M$ be the maximal (with respect to "$<$", cf. 2.5) subset such that $M$ is non-essential and $T_{r,s}(M)v_+$ can not be represented as a linear combination of some $T_{r,s}(J)v_+$ with $J \in \mathbf{E}(r,s)$, $J > M$. Then $T_{r,s}(M)v_+ \neq 0$. So, by Lemma 8.1, there exists $i \in (r,s] \setminus M$ with

$$T_{i,s}(M \cap (i,s])v_+ = 0. \tag{72}$$

By Lemma 7.8, we have

$$\begin{aligned} &T_{r,i-1}(M \cap (r,i))T_{i,s}(M \cap (i,s]) \\ &\quad = T_{r,s}(M) - T_{r,s}\Big(\Big(M \setminus \{\max(M \cap (r,i))\}\Big) \cup \{i\}\Big) \\ &\quad - \mathcal{C}\Big(\max(M \cap (r,i)), i\Big) T_{r,s}(M \cup \{i\}), \end{aligned}$$

where we omit the second summand in the right hand side and interpret the value $\max(M \cap (r,i))$ as $r$ if $M \cap (r,i) = \emptyset$. Applying both sides of this equation to $v_+$ and using (72), we get

$$\begin{aligned} T_{r,s}(M)v_+ &= T_{r,s}\Big(\Big(M \setminus \{\max(M \cap (r,i))\}\Big) \cup \{i\}\Big)v_+ \\ &\quad + \mathcal{C}\Big(\max(M \cap (r,i)), i\Big) T_{r,s}(M \cup \{i\})v_+. \end{aligned}$$

It now remains to note that

$$\Big(M \setminus \{\max\left(M \cap (r, i)\right)\}\Big) \cup \{i\} > M$$

since $i > \max\left(M \cap (r, i)\right)$, $i \notin M$, and

$$M \cup \{i\} > M$$

since $i \notin M$. □

**Corollary 8.3.** *Let $\mu = \lambda - \alpha(r, s)$. Then the set (71) generates $L(\lambda)_\mu$ as a vector space.*

*Proof.* Since any $T_{r,s}(J) \in \mathcal{U}^-_{-\alpha(r,s)}$ we get that all $T_{r,s}(J)v_+ \in L(\lambda)_\mu$. In view of Lemmas 7.1 and 7.7(ii), it suffices to prove that any vector $T_{r,s}(M)v_+$ with $M \subseteq (r, s]$ is a linear combination of vectors from the set (71).

If $L(\lambda)_\mu = 0$ then any $T_{r,s}(M)v_+ = 0$, hence all $M \subseteq (r, s]$ are non-essential by Lemma 8.1. We interpret the vector space generated by the empty set of vectors as 0.

So we may assume $L(\lambda)_\mu \neq 0$. Now it suffices to apply Lemma 8.2. □

Recall the set $\hat{J}$ defined for any $J \subseteq (r, s]$ in 2.8, and constants $h_{r,s}(I, J)$ defined in Section 7 before Lemma 7.11.

**Lemma 8.4.** *If $T_{r,s}(J)v_+ \neq 0$ then $h_{r,s}(\hat{J}, J) \neq 0$.*

*Proof.* Let

$$J = [\beta_1, \gamma_1] \cup [\beta_2, \gamma_2] \cup \cdots \cup [\beta_N, \gamma_N],$$
$$\hat{J} = [b_1, c_1] \cup [b_2, c_2] \cup \cdots \cup [b_H, c_H]$$

be decompositions into the connected components, cf. Definition 1.7. Then, by Definitions 2.8, 2.7, for all $v = 1, 2, \ldots, H$ there exist $u_v \in [1, N]$ such that $b_v = \beta_{u_v}$, $c_v \leq \gamma_{u_v}$.

By Lemma 7.13,

$$h_{r,s}(\hat{J}, J) = \Big[\prod_{d_0 \in [r, b_1 - 2] \setminus \Omega_0(\hat{J}, J)} \mathcal{B}(r, d_0)\Big]$$
$$\times \Big[\prod_{t=1}^{H-1} \prod_{d_t \in [c_t, b_{t+1} - 2] \setminus \Omega_t(\hat{J}, J)} \mathcal{B}(c_t, d_t)\Big]\Big[\prod_{d_H \in [c_H, s] \setminus \Omega_H(\hat{J}, J)} \mathcal{B}(c_H, d_H)\Big].$$

If $h_{r,s}(\hat{J}, J) = 0$ then some of the factors in this product is zero. If $\mathcal{B}(r, d_0) \equiv 0$ for some $d_0 \in [r, b_1 - 2] \setminus \Omega_0(\hat{J}, J)$ then $T_{r,s}(J)v_+ = 0$ by Lemma 7.9 (we can take $u = b_1 - 1, d = d_0$ in that lemma). Let some $\mathcal{B}(c_t, d_t) = 0$ for $t \in [1, H]$, $d_t \in [c_t, b_{t+1} - 2] \setminus \Omega_t(\hat{J}, J)$ (we interpret $b_{H+1}$ as $s + 2$). Then $\hat{J} \setminus \{c_t\}$ is linked with $J$, which contradicts the definition of $\hat{J}$, cf. 2.8. □

**Lemma 8.5.** *Let $T_{r,s}(J)v_+ \neq 0$. Then $\hat{J}$ is maximal among the subsets $I \subseteq (r, s]$ such that $h_{r,s}(I, J) \neq 0$.*

*Proof.* In view of Lemma 8.4 we have only to prove that $h_{r,s}(I,J)=0$ for any $I>\hat{J}$. Assume on the contrary that $I>\hat{J}$ and $h_{r,s}(I,J)\neq 0$. By Lemma 7.11 we have

$$I \ll J. \tag{73}$$

Let

$$\begin{aligned}I&=[x_1,y_1]\cup[x_2,y_2]\cup\cdots\cup[x_G,y_G],\\ \hat{J}&=[b_1,c_1]\cup[b_2,c_2]\cup\cdots\cup[b_H,c_H],\\ J&=[\beta_1,\gamma_1]\cup[\beta_2,\gamma_2]\cup\cdots\cup[\beta_N,\gamma_N]\end{aligned}$$

be decompositions into connected components. Since $\hat{J}\ll J$, Definition 2.7(i) yields

$$b_v=\beta_{u_v} \quad \text{for some} \quad u_v\in[1,N],\ v=1,2,\dots,H. \tag{74}$$

By Lemma 7.12,

$$h_{r,s}(I,J)=\pm\prod_{v=0}^{G}\xi_{y_v,x_{v+1}-2}(\Delta_t(I,J)), \tag{75}$$

where $y_0$ and $x_{G+1}$ are interpreted as $r$ and $s+2$, respectively. Let $i$ be the maximal integer with $i\in I$, $i\notin\hat{J}$. Such $i$ is well defined since $I>\hat{J}$ implies $I\not\subseteq\hat{J}$. There exists $t\in[1,G]$ with $i\in[x_t,y_t]$. We claim that

$$i=y_t. \tag{76}$$

Indeed, otherwise $\emptyset\neq(i,y_t]\subseteq\hat{J}$, $i\notin\hat{J}$ imply $i+1=b_v$ for some $v\in[1,H]$, i.e. $i+1=\beta_{u_v}$ for some $u_v\in[1,N]$ by (74). Hence $i\notin J$. In particular, $I\not\subseteq J$ which contradicts (73).

We shall prove that

$$\xi_{y_t,x_{t+1}-2}(\Delta_t(I,J))=0. \tag{77}$$

Then (75) will imply $h_{r,s}(I,J)=0$ giving a contradiction which proves our lemma.

To prove (77), note that the choice of $i$ yields $I\cap(i,s]\subseteq\hat{J}$. Since $\hat{J}<I$ this implies $I\cap(i,s]=\hat{J}\cap(i,s]$. Since $i=y_t$ by (76), we have $I\cap(i,s]=\cup_{v=t+1}^{G}[x_v,y_v]$. Now $i\notin\hat{J}$ implies

$$\hat{J}\cap[y_t,s]=\cup_{v=t+1}^{G}[x_v,y_v]. \tag{78}$$

Therefore

$$[x_{G-v},y_{G-v}]=[b_{H-v},c_{H-v}],\quad v=0,1,\dots,G-(t+1). \tag{79}$$

Put $w=H-G+(t+1)$. Then

$$x_{t+1}=b_w. \tag{80}$$

Indeed, if $t<G$ this follows by putting $v=G-(t+1)$ in (79). If $t=G$ (i.e. $\hat{J}\cap[y_t,s]=\emptyset$) then we interpret $x_{G+1}=s+2=b_{H+1}$.

Note that $y_t \le x_{t+1} - 2 = b_w - 2$, cf. (80). On the other hand, $y_t > c_{w-1}$ since, in view of (78)-(80), we have

$$[y_t, s] \cap \hat{J} = \cup_{v=t+1}^{G} [x_v, y_v] = \cup_{v=w}^{H} [b_v, c_v].$$

Thus,

$$y_t \in (c_{w-1}, b_w - 2]. \tag{81}$$

Taking into account $y_t \in J$ we conclude that

$$y_t \in \Delta_{w-1}(\hat{J}, J). \tag{82}$$

We also have $\Delta_t(I, J) \subseteq (y_t, x_{t+1} - 2] \subseteq (c_{w-1}, b_w - 2]$ in view of (81) and (80). On the other hand, $\Delta_t(I, J) \subseteq J$. Summarizing, $\Delta_t(I, J) \subseteq \Delta_{w-1}(\hat{J}, J)$. Together with (82) this means that

$$\{y_t\} \sqcup \Delta_t(I, J) \subseteq \Delta_{w-1}(\hat{J}, J). \tag{83}$$

Since $\hat{J}$ is linked with $J$ then, by Definition 2.7, for all $j \in \Delta_{w-1}(\hat{J}, J)$ there exist $d(j) \in [j, b_w - 2] = [j, x_{t+1} - 2]$ (cf. (80)), all distinct, such that $\mathcal{B}(j, d(j)) \equiv 0$. So (83) implies that for all $j \in \{y_t\} \sqcup \Delta_t(I, J)$ there exist $d(j) \in [j, x_{t+1} - 2]$, all distinct, such that $\mathcal{B}(j, d(j)) \equiv 0$. Put

$$D = \{d(j) \mid j \in \Delta_t(I, J)\}.$$

Then by Lemma 7.10 we have

$$\xi_{y_t, x_{t+1}-2}(\Delta_t(I, J)) = \prod_{v \in [y_t, x_{t+1}-2] \setminus D} \mathcal{B}(y_t, v).$$

However, the last product is zero since $\mathcal{B}(y_t, d(y_t)) = 0$, and $d(y_t) \in [y_t, x_{t+1} - 2] \setminus D$. This proves (77) and the lemma. $\square$

**Theorem 8.6.** *Let $\mu = \lambda - \alpha(r, s)$. Then*

$$\{T_{r,s}(J)v_+ \mid J \text{ is essential in } (r, s]\}$$

*is a basis of $L(\lambda)_\mu$.*

*Proof.* In view of Corollary 8.3, we have only to prove that vectors of the set (71) are linearly independent. Assume that there exists a non-trivial linear combination of the form

$$\sum \{c_J T_{r,s}(J)v_+ \mid J \in \mathbf{E}(r, s)\} = 0.$$

Since all $\hat{J}$ for $J \in \mathbf{E}(r, s)$ are distinct by virtue of Lemma 2.9, we may choose $M \subseteq (r, s]$ such that $c_M \neq 0$, and $\hat{J} > \hat{M}$, $J \in \mathbf{E}(r, s)$ imply $c_J = 0$. Then by Lemma 8.5, we have

$$Q_{r,s}(\hat{M}) T_{r,s}(J) v_+ = h_{r,s}(\hat{M}, J) v_+ = 0$$

for any $J$ with $J \neq M$, $c_J \neq 0$. So

$$0 = Q_{r,s}(\hat{M}) \sum \{c_J T_{r,s}(J)v_+ \mid J \in \mathbf{E}(r, s)\} = c_M h_{r,s}(\hat{M}, M) v_+ \neq 0,$$

where the last inequality follows from Lemma 8.4, because $T_{r,s}(M)v_+ \neq 0$ for any essential $M$ by Lemma 8.1. $\square$

We have proved that (71) is a basis of $L(\lambda)_{\lambda-\alpha(r,s)}$. However the operators $T_{r,s}(J)$ are quite difficult to compute. Actually they are defined using an inductive procedure (cf. [14, 2.5,2.3]). Our next goal is to construct a "PBW-type" basis in $L(\lambda)_{\lambda-\alpha(r,s)}$, i.e. a basis consisting of vectors of the form

$$Y_{r,j_1-1}Y_{j_1,j_2-1}\dots Y_{j_f,s}v_+.$$

Elements of this new basis will also be parametrized by the essential subsets. The matrix relating the two bases is uni-triangular (see Lemma 7.7(i)).

**Definition 8.7.** For $J \subseteq (r,s]$ we define a subset $\bar{J} = (r,s] \setminus J$.

**Theorem 8.8.** *Let $\mu = \lambda - \alpha(r,s)$. Then*

$$\{Y_{r,s}(\bar{J})v_+ \mid J \subseteq (r,s] \text{ is essential}\} \tag{84}$$

*is a basis of $L(\lambda)_\mu$.*

*Proof.* It follows from Theorem 8.6 that $\dim L(\lambda)_\mu = \varepsilon(r,s)$, cf. 1.13. So it suffices to prove that the set (84) generates $L(\lambda)_\mu$. Now, by Lemmas 7.1 and 7.6, it is enough to show that any $Y_{r,s}(M)v_+$ belongs to the vector space $V$ generated by (84).

Assume not. Let $M$ be a maximal subset in $(r,s]$ such that $Y_{r,s}(\bar{M})v_+ \notin V$. Then $M$ is non-essential. By Lemma 8.2 we have

$$T_{r,s}(M)v_+ = \sum\{c_J T_{r,s}(J)v_+ \mid J \in \mathbf{E}(r,s),\ J > M\}. \tag{85}$$

Lemma 7.7(i) implies that

$$T_{r,s}(I)v_+ = Y_{r,s}(\bar{I})v_+ + \sum_{F<\bar{I}} \mathrm{const}_F\, Y_{r,s}(F)v_+ \tag{86}$$

for any $I \subseteq (r,s]$.

By Lemma 2.6, $J > M$ implies $\bar{J} < \bar{M}$, so $F < \bar{J}$ implies $F < \bar{M}$. Therefore it follows from (85) and (86) that $Y_{r,s}(\bar{M})v_+$ is a linear combination of some $Y_{r,s}(F)v_+$ with $F < \bar{M}$. It follows from the choice of $M$ that any $Y_{r,s}(F)v_+$ with $F < \bar{M}$ belongs to $V$. Hence $Y_{r,s}(\bar{M})v_+ \in V$ giving a contradiction with the choice of $M$. $\square$

We now want to describe one more "PBW-type" basis of $L(\lambda)_{\lambda-\alpha(r,s)}$. The elements of this basis will be parametrized by the $\sigma$-subsets in $(r,s]$, cf. 4.1, 5.6(ii).

**Theorem 8.9.** *Let $\mu = \lambda - \alpha(r,s)$. Then the set of vectors*

$$\{Y_{r,s}(J)v_+ \mid J \subseteq (r,s] \text{ is a } \sigma\text{-subset}\} \tag{87}$$

*is a basis of $L(\lambda)_\mu$.*

*Proof.* By Corollary 5.6(ii), the number of the $\sigma$-subsets in $(r,s]$ is equal to the number of the essential subsets in $(r,s]$. So, in view of Theorem 8.6, we have only to prove that the set (87) generates $L(\lambda)_\mu$. By Lemmas 7.1 and 7.6, it is enough to show that any $Y_{r,s}(M)v_+$ with $M \subseteq (r,s]$ belongs to the vector space $V$ generated by (87).

Let

$$I = \{i_1 < i_2 < \cdots < i_f\}$$

be the maximal subset of $(r,s]$ such that

$$Y_{r,s}(I)v_+ = Y_{r,i_1-1}Y_{i_1,i_2-1}\dots Y_{i_{f-1},i_f-1}Y_{i_f,s}v_+ \notin V.$$

Then $Y_{r,s}(I)v_+ \neq 0$, and $I$ is not a $\sigma$-subset, i.e.

$$\sigma(I) = \sigma(r,i_1-2)\sigma(i_1,i_2-2)\dots\sigma(i_{f-1},i_f-2)\sigma(i_f,s) = 0.$$

Assume $\sigma(i_f,s)=0$. There exists $i \in [i_f,s]$ with $\lambda_i \not\equiv 0$, since otherwise $Y_{i_f,s}v_+ = 0$ and hence $Y_{r,s}(I)v_+ = 0$ giving a contradiction. Now, by Lemma 7.15,

$$Y_{i_f,s}v_+ = \sum_{J\subseteq(i_f,s],J\neq\emptyset} c_J Y_{i_f,s}(J)v_+.$$

Hence

$$Y_{r,s}(I)v_+ = \sum_{J\subseteq(i_f,s],J\neq\emptyset} c_J Y_{r,s}(I\cup J)v_+.$$

Note that $J \subseteq (i_f,s]$, $J \neq \emptyset$ imply $I \cup J > I$. So any $Y_{r,s}(I\cup J)v_+ \in V$. Therefore $Y_{r,s}(I)v_+ \in V$ giving a contradiction.

Thus we may assume that there exists $t \in [0,f-1]$ with $\sigma(i_t,i_{t+1}-2)=0$ (here $i_0$ is interpreted as $r$). In particular, $i_t \le i_{t+1}-2$ (cf. Definition 1.16). Denote

$$\begin{aligned}
I_1 &= \{i_1,i_2,\dots,i_{t-1}\},\\
I_2 &= \{i_{t+2},i_{t+3},\dots,i_f\},\\
i &= i_t,\\
j &= i_{t+1}-1.
\end{aligned}$$

Then $\sigma(i,j-1)=0$, and

$$Y_{r,s}(I) = Y_{r,i-1}(I_1)Y_{i,j}Y_{j+1,s}(I_2).$$

Assume first that $i=j-1$. Then $\sigma(i,j-1)=0$ implies $\lambda_i \equiv 0$, hence $Y_{i,i}v_+ = 0$. So using elementary commutation relations in $\mathcal{U}_K$ (see [14, 1.2]) we get

$$\begin{aligned}
Y_{i,j}Y_{j+1,s}(I_2)v_+ &= Y_{i,i+1}Y_{i+2,s}(I_2)v_+\\
&= (-Y_{i,i}Y_{i+1,i+1} + Y_{i+1,i+1}Y_{i,i})Y_{i+2,s}(I_2)v_+\\
&= -Y_{i,i}Y_{i+1,i+1}Y_{i+2,s}(I_2)v_+ + Y_{i+1,i+1}Y_{i+2,s}(I_2)Y_{i,i}v_+\\
&= -Y_{i,i}Y_{i+1,i+1}Y_{i+2,s}(I_2)v_+.
\end{aligned}$$

Therefore

$$Y_{r,s}(I)v_+ = -Y_{r,i-1}(I_1)Y_{i,i}Y_{i+1,i+1}Y_{i+2,s}(I_2)v_+ = \\ = -Y_{r,s}(I \cup \{i+1\})v_+.$$

Note that $i+1 \notin I$, so $I \cup \{i+1\} > I$. Therefore $Y_{r,s}(I \cup \{i+1\})v_+ \in V$ by the choice of $I$, giving a contradiction.

Now assume that $i \leq j-2$. To apply Lemma 7.15, we make use of a commutation trick from the proof of Proposition 4.1 in [12].

By elementary commutation relations in $\mathcal{U}_K$ (see [14, 1.2]) we may write

$$\begin{aligned} Y_{i,j}Y_{j+1,s}(I_2)v_+ &= (-Y_{i,j-1}Y_{j,j} + Y_{j,j}Y_{i,j-1})Y_{j+1,s}(I_2)v_+ \\ &= -Y_{i,j-1}Y_{j,j}Y_{j+1,s}(I_2)v_+ + Y_{j,j}Y_{j+1,s}(I_2)Y_{i,j-1}v_+. \end{aligned}$$

If $\lambda_u \equiv 0$ for all $u \in [i, j-1]$ then we get $Y_{i,j-1}v_+ = 0$ so

$$Y_{i,j}Y_{j+1,s}(I_2)v_+ = -Y_{i,j-1}Y_{j,j}Y_{j+1,s}(I_2)v_+.$$

Hence

$$Y_{r,s}(I)v_+ = -Y_{r,i-1}(I_1)Y_{i,j-1}Y_{j,j}Y_{j+1,s}(I_2)v_+ = \\ = -Y_{r,s}(I \cup \{j\})v_+.$$

Since $j \notin I$ we get $I \cup \{j\} > I$, hence $Y_{r,s}(I \cup \{j\})v_+ \in V$ giving a contradiction.

Now assume there is $u \in [i, j-1]$ with $\lambda_u \not\equiv 0$. Then, in view of Lemma 7.15, the equality $\sigma(i, j-1) = 0$ implies

$$Y_{i,j-1}v_+ = \sum_{F \subseteq (i,j-1], F \neq \emptyset} Y_{i,j-1}(F)v_+.$$

So

$$\begin{aligned} Y_{j,j}Y_{j+1,s}(I_2)Y_{i,j-1}v_+ &= Y_{j,j}Y_{j+1,s}(I_2) \sum_{F \subseteq (i,j-1], F \neq \emptyset} Y_{i,j-1}(F)v_+ \\ &= \sum_{F \subseteq (i,j-1], F \neq \emptyset} Y_{j,j}Y_{i,j-1}(F)Y_{j+1,s}(I_2)v_+ \\ &= \sum_{F \subseteq (i,j-1], F \neq \emptyset} \Big(Y_{i,j}(F)Y_{j+1,s}(I_2) + Y_{i,j-1}(F)Y_{j,j}Y_{j+1,s}(I_2)\Big)v_+. \end{aligned}$$

Therefore

$$\begin{aligned} Y_{i,j}Y_{j+1,s}(I_2)v_+ = &-Y_{i,j-1}Y_{j,j}Y_{j+1,s}(I_2)v_+ \\ &+ \sum_{F \subseteq (i,j-1], F \neq \emptyset} \Big(Y_{i,j}(F)Y_{j+1,s}(I_2) + Y_{i,j-1}(F)Y_{j,j}Y_{j+1,s}(I_2)\Big)v_+. \end{aligned}$$

Hence

$$Y_{r,s}(I)v_+ = Y_{r,i-1}(I_1)Y_{i,j}Y_{j+1,s}(I_2)v_+ =$$
$$= -Y_{r,s}(I\cup\{j\})v_+ + \sum_{F\subseteq(i,j-1],F\neq\emptyset}\Big(Y_{r,s}(I\cup F) - Y_{r,s}(I\cup\{j\}\cup F)\Big)v_+.$$

Note that $j\notin I$ implies $I\cup\{j\}>I$, and $F\subseteq(i,j-1]$, $F\neq\emptyset$ imply $I\cup F>I$, $I\cup\{j\}\cup F>I$. So, by the choice of $I$, the right hand side of the last equation belongs to $V$, hence $Y_{r,s}(I)v_+$ belongs to $V$, giving a contradiction. □

As a consequence of Theorems 8.6 and 8.9 we get an enumerative description of the weight multiplicity $\dim L(\lambda)_{\lambda-\alpha(r,s)}$ as a number of essential subsets or a number of $\sigma$-subsets. Using combinatorial results obtained in Part I we can deduce recurrence formulas for this weight multiplicity.

Recall the functions $\nu,\nu'$ defined in 3.1.

**Corollary 8.10.** (i) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \sharp\{$*essential subsets in* $(r,s]\}$;

(ii) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \sharp\{\sigma$*-subsets in* $(r,s]\}$;

(iii) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \nu(r,s)+\sum_{i=r+1}^{s}\nu'(r,i-1)\dim L(\lambda)_{\lambda-\alpha(i,s)}$;

(iv) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \sigma(r,s)+\sum_{i=r+1}^{s}\sigma(r,i-2)\dim L(\lambda)_{\lambda-\alpha(i,s)}$;

(v) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \sum_{J\subseteq(r,s]}\nu(J)$;

(vi) $\dim L(\lambda)_{\lambda-\alpha(r,s)} = \sum_{J\subseteq(r,s]}\sigma(J)$;

(vii) *Let*

$$I=\{i_1<i_2<\dots<i_k\}=\{i\in[r,s]\mid \lambda_i\not\equiv 0\}.$$

*If* $I=\emptyset$ *then* $\dim L(\lambda)_{\lambda-\alpha(r,s)}=0$. *Otherwise*

$$\begin{aligned}\dim L(\lambda)_{\lambda-\alpha(r,s)} &= \dim L(\lambda)_{\lambda-\alpha(i_1,i_k)}\\ &= \sigma(i_1,i_k)+(i_2-i_1)\dim L(\lambda)_{\lambda-\alpha(i_2,i_k)}\\ &+\sum_{t=3}^{k}\Big((i_t-i_{t-1}-1)\sigma(i_1,i_{t-1})+\sigma(i_1,i_{t-2})\Big)\dim L(\lambda)_{\lambda-\alpha(i_t,i_k)}.\end{aligned}$$

*Proof.* (i) and (ii) follow from Theorems 8.6 and 8.9. Now (iii) follows from Corollary 5.6(iv), (iv) follows from Proposition 5.5, (v) follows from Corollary 5.6(iii), and (vi) follows from Corollary 5.6(i). Finally, part (vii) follows from Lemma 2.3(i) and Corollary 5.9. □

We illustrate how the formula 8.10(vii) works (we could also use 8.10(iii) instead of it to prove Proposition 8.11 below). Recall the "generalized Fibonacci numbers" $f(i_1,i_2,\dots,i_k)$ defined in 1.21.

**Proposition 8.11.** *Assume that*

$$\{i_1<i_2<\dots<i_k\}=\{i\in[r,s]\mid\lambda_i\not\equiv 0\}\neq\emptyset.$$

*Then*

$$\dim L(\lambda)_{\lambda-\alpha(r,s)}\geq f(i_1,i_2,\dots,i_k).$$

*Moreover,*

$$\dim L(\lambda)_{\lambda-\alpha(r,s)} = f(i_1, i_2, \dots, i_k)$$

*if and only if*

$$\mathcal{B}(i_j, i_{j+1}) \equiv 0 \textit{ for all } j = 1, 2, \dots, k-1. \tag{88}$$

*Proof.* Put $d(r,s) = \dim L(\lambda)_{\lambda-\alpha(r,s)}$. We prove the proposition by induction on $k$. First note that $d(r,s) = d(i_1, i_k)$ in view of 8.10(vii).

Let $k = 1$. Then we have $d(r,s) = d(i_1, i_1) = 1 = f(i_1)$, as required. Let $k = 2$. Then, by Corollary 8.10(vii), we have

$$d(r,s) = \sigma(i_1, i_2) + (i_2 - i_1)d(i_2, i_2) = \sigma(i_1, i_2) + (i_2 - i_1).$$

In view of Lemma 3.3, we have

$$\sigma(i_1, i_2) = \begin{cases} 1, & \text{if } \mathcal{B}(i_1, i_2) \not\equiv 0; \\ 0, & \text{if } \mathcal{B}(i_1, i_2) \equiv 0. \end{cases}$$

Thus for $k = 2$, $d(r,s)$ takes its minimal value if and only if (88) holds, and this value is equal to $i_2 - i_1 = f(i_1, i_2)$.

Let $k > 2$, and assume the proposition has been proved for all $k' < k$. By 8.10(vii) we have

$$\begin{aligned} d(r,s) =& d(i_1, i_k) = \sigma(i_1, i_k) + (i_2 - i_1)d(i_2, i_k) \\ &+ \sum_{t=3}^{k} \Big((i_t - i_{t-1})\sigma(i_1, i_{t-1}) + \sigma(i_1, i_{t-2})\Big) d(i_t, i_k) \\ =& \sigma(i_1, i_k) + (i_2 - i_1)d(i_2, i_k) + \Big((i_3 - i_2)\sigma(i_1, i_2) + \sigma(i_1, i_1)\Big) d(i_3, i_k) \\ &+ \sum_{t=4}^{k} \Big((i_t - i_{t-1})\sigma(i_1, i_{t-1}) + \sigma(i_1, i_{t-2})\Big) d(i_t, i_k). \end{aligned}$$

Since $\sigma(i_1, i_1) = 1$ we get

$$\begin{aligned} d(r,s) =& (i_2 - i_1)d(i_2, i_k) + d(i_3, i_k) + \sigma(i_1, i_k) + (i_3 - i_2)\sigma(i_1, i_2)d(i_3, i_k) \\ &+ \sum_{t=4}^{k} \Big((i_t - i_{y-1})\sigma(i_1, i_{t-1}) + \sigma(i_1, i_{t-2})\Big) d(i_t, i_k). \end{aligned} \tag{89}$$

By the inductive hypothesis we have

$$d(i_2, i_k) \geq f(i_2, i_3, \dots, i_k) \quad \text{and} \quad d(i_3, i_k) \geq f(i_3, i_4, \dots, i_k).$$

So $d(r,s) \geq f(i_2, i_3, \dots, i_k)$.

On the other hand, if (88) holds then $\sigma(i_1, i_j) = 0$ for $j \geq 2$ by virtue of Lemma 3.3. Using the inductive hypothesis we get

$$\begin{aligned} d(r,s) &= (i_2 - i_1)d(i_2, i_k) + d(i_3, i_k) = (i_2 - i_1)f(i_2, i_3, \dots, i_k) + f(i_3, i_4, \dots, i_k) \\ &= f(i_1, i_2, \dots, i_k). \end{aligned}$$

Finally, again by the inductive hypothesis we obtain

$$\begin{aligned}
&d(i_2,i_k) \ge f(i_2,i_3,\dots,i_k), \quad d(i_3,i_k) \ge f(i_3,i_4,\dots,i_k);\\
&d(i_2,i_k) = f(i_2,i_3,\dots,i_k) \quad \text{if and only if} \quad \mathcal{B}(i_j,i_{j+1}) = 0 \text{ for all } j = 2,3,\dots,k-1;\\
&d(i_3,i_k) = f(i_3,i_4,\dots,i_k) \quad \text{if and only if} \quad \mathcal{B}(i_j,i_{j+1}) = 0 \text{ for all } j = 3,4,\dots,k-1.
\end{aligned}$$

Since all summands in the right hand side of (89) are non-negative this implies that $d(r,s) = f(i_1,i_2,\dots,i_k)$ if and only if $\mathcal{B}(i_j,i_{j+1}) = 0$ for $j = 2,3,\dots,k-1$ and $\sigma(i_1,i_j) = 0$ for $j = 2,3,\dots,k$. By Lemma 3.3, this is equivalent to (88). □

## 9. Some composition multiplicities of $L(\lambda)_1 \mid A_{l-1}(K)$.

Recall that for a weight $\nu = (\nu_1,\nu_2,\dots,\nu_l) \in P(A_l)$ we denote by $\bar\nu$ the weight $\bar\nu = (\nu_1,\nu_2,\dots,\nu_{l-1}) \in P(A_{l-1})$, cf. Section 7. Throughout this section we denote

$$\mu^i = \overline{\lambda - \alpha(i,l)} \in P(A_{l-1}).$$

Note that $\alpha(i,l) = -\omega_{i-1} + \omega_i + \omega_l$ (where $\omega_0$ is interpreted as 0 if $i=1$). Since $\lambda$ is supposed to be dominant throughout Part II, $\mu^i \in P_+(A_{l-1})$ if and only if $\lambda_i > 0$.

**Definition 9.1.** We define the integers $\tau_i$, $i = 1,2,\dots,l$ as follows. Put $\tau_i = 0$ if $\lambda_i = 0$. Let $\lambda_i > 0$. Then define $\tau_i$ as the multiplicity of the irreducible $A_{l-1}(K)$-module $L(\mu^i)$ in $L(\lambda)_1 \mid A_{l-1}(K)$, i.e.

$$\tau_i = \left[L(\lambda)_1 \mid A_{l-1}(K) : L(\mu^i)\right].$$

Recall the integers $\varepsilon(r,s), \varepsilon'(r,s)$ defined in 1.13 and (7).

**Lemma 9.2.** *Let $1 \le i \le l$. Then*

$$\varepsilon(i,l) = \tau_i + \sum_{u=i+1}^{l} \tau_u \varepsilon'(i,u-1).$$

*Proof.* Note that $L(\lambda)_{\lambda-\alpha(i,l)} \subseteq L(\lambda)_1$. In view of Lemma 7.3, this implies

$$L(\lambda)_{\lambda-\alpha(i,l)} = \left(L(\lambda)_1 \mid A_{l-1}(K)\right)_{\mu^i}. \tag{90}$$

Assume that $\eta \in P(A_l)$ is a weight of the first level of $L(\lambda)$, i.e. $L(\lambda)_\eta \subseteq L(\lambda)_1$. Then $\bar\eta \ge \mu^i$ if and only if and only $\eta = \lambda - \sum_{u=i}^{l} n_i\alpha_i$ with $n_l = 1$ and $n_u = 0$ or 1 for $u = i, i+1, \dots, l-1$ (take into account Lemma 7.3). If, moreover, $L(\bar\eta)$ occurs as a composition factor in $L(\lambda) \mid A_{l-1}(K)$ then Lemma 7.4 implies that $\eta = \lambda - \alpha(j,l)$ for some $j \in [i,l]$. So we may write

$$L(\lambda)_1 \mid A_{l-1}(K) \sim \sum_{u=i}^{l} \tau_u L(\mu^u) + (*) \tag{91}$$

where $(*)$ stands for composition factors$L(\mu)$ with $\mu \not\ge \mu^i$ (in $P(A_{l-1})$).

It follows from (91) that

$$\dim\left(L(\lambda)_1 \mid A_{l-1}(K)\right)_{\mu^i} = \sum_{u=i}^{l} \tau_u \dim L(\mu^u)_{\mu^i}. \tag{92}$$

Now, by Corollary 8.10(i), we have

$$\dim L(\lambda)_{\lambda-\alpha(i,l)} = \varepsilon(i,l),$$

so (90) implies

$$\dim \left(L(\lambda)_1 \mid A_{l-1}(K)\right)_{\mu^i} = \varepsilon(i,l).$$

On the other hand, let $u \in (i,l]$. Note that

$$\mu^u = \sum_{w=i}^{u-2} \lambda_w \omega_w + (\lambda_{u-1}+1)\omega_{u-1} + (*)$$

where $(*)$ stands for the summands of the form $c\omega_j$ with $j \notin [i, u-1]$. However $\mu^i = \mu^u - \alpha(i,u-1)$. So it follows from Corollary 8.10(i) that

$$\dim L(\mu^u)_{\mu^i} = \varepsilon'(i,u-1) \quad \text{for} \quad u \in (i,l].$$

Besides, $\dim L(\mu^i)_{\mu^i} = 1$.

Now the lemma follows from (92). □

**Theorem 9.3.** *Let $1 \leq i \leq l$. Then $\tau_i = \nu(i,l)$ (cf. Definition 3.1).*

*Proof.* We have by Corollary 5.8(ii)

$$\nu(i,l) = \varepsilon(i,l) - \sum_{u=i+1}^{l} \nu(u,l)\varepsilon'(i,u-1).$$

However, by Lemma 9.2 we also have

$$\tau_i = \varepsilon(i,l) - \sum_{u=i+1}^{l} \tau_u \varepsilon'(i,u-1).$$

Now the statement is obtained by the reverse induction on $i = l, l-1, \dots, 1$. □

If we consider the subgroup $GL_{n-1} < SL_n$ rather than $SL_{n-1} < SL_n$ then we can speak about multiplicities in the whole $L(\lambda)\downarrow_{GL_{n-1}}$ instead of multiplicities in the first level. More precisely, let $\tilde{A}_{l-1}(K) < A_l(K)$ be the subgroup generated by the subgroup $A_{l-1}(K)$ and the whole Cartan subgroup of $A_l(K)$ (see [13, Section 1] for more details). Then $\tilde{A}_{l-1}(K) \cong GL_l(K)$ is diagonally embedded into $A_l(K) \cong SL_{l+1}(K)$. The Cartan subgroups of $\tilde{A}_{l-1}(K)$ and $A_l(K)$ coincide. So characters of the Cartan subgroup of $\tilde{A}_{l-1}(K)$ can be considered as elements of $P(A_l)$. Then the (rational) irreducible modules over $\tilde{A}_{l-1}(K)$ are parametrized by those $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P(A_l)$ with $\lambda_1, \lambda_2, \dots, \lambda_{l-1} \geq 0$, cf. [13]. The corresponding irreducible module is denoted by $\tilde{L}(\lambda)$. Similarly, we denote the Weyl module over $\tilde{A}_{l-1}(K)$ with highest weight $\lambda$ by $\tilde{V}(\lambda)$. (In [13] we wrote $L(\lambda_1, \dots, \lambda_{l-1}, \{\lambda_l\})$ for $\tilde{L}(\lambda)$ , and $V(\lambda_1, \dots, \lambda_{l-1}, \{\lambda_l\})$ for $\tilde{V}(\lambda)$).

Of course, part (i) of the following Corollary is not new.

**Corollary 9.4.** (i) *Assume that* $\mu = \lambda - \sum_{i=1}^{l-1} n_i\alpha_i$, *and* $\tilde{L}(\mu)$ *occurs as a composition factor of the restriction* $L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$. *Then* $\mu = \lambda$, *and*

$$\Big[L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)} \colon \tilde{L}(\lambda)\Big] = 1.$$

(ii) *Assume that* $\mu = \lambda - \alpha_l - \sum_{i=1}^{l-1} n_i\alpha_i$, *with* $n_i \leq 1$, $i = 1, 2, \ldots, l-1$. *If* $\tilde{L}(\mu)$ *occurs as a composition factor of the restriction* $L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$ *then* $\mu = \lambda - \alpha(j, l)$, *and*

$$\Big[L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)} \colon \tilde{L}(\lambda - \alpha(j, l))\Big] = \nu(j, l).$$

*Proof.* It is proved in [22] that

$$\bigoplus \{L_\nu \mid \nu \text{ has a form } \lambda - \sum_{i=1}^{l-1} m_i\alpha_i\}$$

is an irreducible $\tilde{A}_{l-1}(K)$-submodule of $L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$ isomorphic to $\tilde{L}(\lambda)$. Moreover, $\dim L(\lambda)_\lambda = 1$ implies

$$\Big[L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)} \colon \tilde{L}(\lambda)\Big] = 1.$$

This proves (i). Part (ii) follows from Lemmas 7.4, 7.3 and Theorem 9.3. □

## 10. Decomposition numbers $[V(\lambda) : L(\lambda - \alpha)]$.

Throughout the section we denote

$$\kappa(r, s) = \big[V(\lambda) : L(\lambda - \alpha(r, s))\big],$$
$$\kappa'(r, s) = \big[V(\lambda + \omega_s) : L(\lambda + \omega_s - \alpha(r, s))\big],$$
$$z_i = \begin{cases} 1, & \text{if } \lambda_i > 0, \\ 0, & \text{if } \lambda_i = 0. \end{cases}$$

The following result is well known.

**Lemma 10.1.** *If* $\lambda_r > 0$, $\lambda_r \equiv 0$ *then* $[V(\lambda) : L(\lambda - \alpha_r)] = 1$. *Otherwise we have* $[V(\lambda) : L(\lambda - \alpha_r)] = 0$.

*Proof.* One has

$$\dim L(\lambda)_{\lambda-\alpha_r} = \begin{cases} 1, & \text{if } \lambda_r \not\equiv 0 \\ 0, & \text{if } \lambda_r \equiv 0. \end{cases}$$

Moreover,

$$\dim V(\lambda)_{\lambda-\alpha_r} = \begin{cases} 1, & \text{if } \lambda_r > 0 \\ 0, & \text{if } \lambda_r = 0. \end{cases}$$

Now, the result follows from the equality

$$[V(\lambda) : L(\lambda - \alpha_r)] = \dim V(\lambda)_{\lambda-\alpha_r} - \dim L(\lambda)_{\lambda-\alpha_r}.$$

□

**Lemma 10.2.** [12, 1.7] *Assume that* $[V(\lambda) : L(\lambda - \alpha(r,s))] \neq 0$. *Then we have* $\lambda_r > 0,\ \lambda_s > 0,\ \mathcal{B}(r,s) \equiv 0$.

*Proof.* Since $\alpha(r,s) = -\omega_{r-1} + \omega_r + \omega_s - \omega_{s+1}$, we must have $\lambda_r > 0, \lambda_s > 0$ for $\lambda - \alpha(r,s)$ to be dominant. The equality $\mathcal{B}(r,s) \equiv 0$ follows from the strong linkage principle, see [12, 1.7] for more details. $\square$

Recall the constants $\bar{\nu}(i,j)$ defined in 3.1.

**Proposition 10.3.** *Assume* $\lambda_1 > 0$. *Then*

$$\kappa(1,l) = 1 - \nu(1,l) + \sum_{i=1}^{l-1} z_{i+1}\kappa'(1,i) - \sum_{i=1}^{l-1} \bar{\nu}(i+1,l)\kappa(1,i). \tag{93}$$

*Proof.* We consider the multiplicity

$$\Big[V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}: \tilde{L}(\lambda - \alpha(1,l))\Big]$$

of the irreducible $\tilde{A}_{l-1}(K)$-module $\tilde{L}(\lambda - \alpha(1,l))$ in the restriction $V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$.

Put

$$V(\lambda)_1 = \bigoplus \Big\{ V(\lambda)_\nu \mid \nu \text{ has a form } \lambda - \alpha_l - \sum_{i=1}^{l-1} m_i\alpha_i \Big\}.$$

Then $V(\lambda)_1$ is a direct summand of $V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$. We denote it by $V(\lambda)_1 \mid \tilde{A}_{l-1}(K)$ in order to underline that $V(\lambda)_1$ is considered as an $\tilde{A}_{l-1}(K)$-module. Clearly,

$$\Big[V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}: \tilde{L}(\lambda - \alpha(1,l))\Big] = \Big[V(\lambda)_1 \mid \tilde{A}_{l-1}(K) : \tilde{L}(\lambda - \alpha(1,l))\Big].$$

It follows from [4], [19] (cf. [12, 1.5] for details) that $V(\lambda)_1 \mid \tilde{A}_{l-1}(K)$ has a filtration whose factors are

$$\Big\{ \tilde{V}(\lambda - \alpha(i,l)) \mid 1 \le i \le l,\ \lambda_i > 0 \Big\}.$$

Let $2 \le i \le l$. Then

$$\lambda - \alpha(i,l) = \lambda_1\omega_1 + \cdots + \lambda_{i-2}\omega_{i-2} + (\lambda_{i-1}+1)\omega_{i-1} + (*)$$

where $(*) \in \mathbb{Z}\omega_i + \cdots + \mathbb{Z}\omega_l$. So each of the $\tilde{V}(\lambda - \alpha(i,l))$, with $2 \le i \le l,\ \lambda_i > 0$ contains a composition factor

$$\tilde{L}(\lambda - \alpha(1,l)) = \tilde{L}(\lambda - \alpha(i,l) - \alpha(1,i-1))$$

with multiplicity $\kappa'(1,i-1)$ by definition of $\kappa$ (take into account [22]).

Moreover, $\tilde{V}(\lambda - \alpha(1,l))$ contains $\tilde{L}(\lambda - \alpha(1,l))$ with multiplicity 1. Thus,

$$\Big[V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}: \tilde{L}(\lambda - \alpha(1,l))\Big] = 1 + \sum_{i=2}^{l} z_i\kappa'(1,i-1). \tag{94}$$

On the other hand, let

$$0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(N)} = V(\lambda)$$

be a composition series of the $A_l(K)$-module $V(\lambda)$. Let $\mu \in P_+(A_l)$ and suppose $L(\mu)$ occurs as a factor of this composition series exactly $x$ times. If $x > 0$ then $\mu \leq \lambda$. Moreover, if $\tilde{L}(\lambda - \alpha(1,l))$ occurs $y$ times as a composition factor of the restriction $L(\mu)\downarrow_{\tilde{A}_{l-1}(K)}$ then $\mu$ contributes $xy$ factors $\tilde{L}(\lambda - \alpha(1,l))$ into $V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)}$. In view of Corollary 9.4, $\mu \leq \lambda$ and $y > 0$ imply $\mu = \lambda$ or $\mu = \lambda - \alpha(1,i)$ for some $i \in [1,l]$.

Assume $\mu = \lambda$. Then $x = [V(\lambda) : L(\lambda)] = 1$. Furthermore,

$$y = \left[L(\lambda)\downarrow_{\tilde{A}_{l-1}(K)} : \tilde{L}(\lambda - \alpha(1,l))\right] = \nu(1,l)$$

by Corollary 9.4(ii). Thus $\mu = \lambda$ contributes $\nu(1,l)$ factors.

Let $\mu = \lambda - \alpha(1,i)$, $1 \leq i \leq l$. Then

$$x = [V(\lambda) : L(\lambda - \alpha(1,i))] = \kappa(1,i).$$

Moreover, if $i = l$ then

$$y = \left[L(\lambda - \alpha(1,l))\downarrow_{\tilde{A}_{l-1}(K)} : \tilde{L}(\lambda - \alpha(1,l))\right] = 1$$

by Corollary 9.4(i). Assume now that $i \in [1,l)$. Then

$$\mu = \lambda - \alpha(1,i) = (*) + (\lambda_{i+1} + 1)\omega_{i+1} + \lambda_{i+2}\omega_{i+2} + \cdots + \lambda_l\omega_l$$

where $(*) \in \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_i$. Therefore

$$y = \left[L(\mu)\downarrow_{\tilde{A}_{l-1}(K)} : \tilde{L}(\lambda - \alpha(1,l))\right] = \left[L(\mu)\downarrow_{\tilde{A}_{l-1}(K)} : \tilde{L}(\mu - \alpha(i+1,l))\right] = \bar{\nu}(i+1,l)$$

by Corollary 9.4(ii).

Summarizing we have

$$\left[V(\lambda)\downarrow_{\tilde{A}_{l-1}(K)} : \tilde{L}(\lambda - \alpha(1,l))\right] = \kappa(1,l) + \nu(1,l) + \sum_{i=1}^{l-1} \kappa(1,i)\bar{\nu}(i+1,l).$$

Comparing this with (94) we get (93). □

From Proposition 10.3 we immediately get (taking into account [22])

**Corollary 10.4.** *Assume $\lambda_r > 0$. Then*

$$\kappa(r,s) = 1 - \nu(r,s) + \sum_{i=r}^{s-1} z_{i+1}\kappa'(r,i) - \sum_{i=r}^{s-1} \bar{\nu}(i+1,s)\kappa(r,i).$$

Recall the constants $\bar{\sigma}(i,j)$ defined in (31). The next result provides a recurrence formula for the decomposition numbers $\kappa(r,s) = [V(\lambda) : L(\lambda - \alpha(r,s))]$. In view of Lemmas 10.1 and 10.2, the assumptions are not very restrictive.

**Corollary 10.5.** *Let* $1 \leq r < s \leq l$, $\lambda_r > 0$, $\lambda_s > 0$, $\mathcal{B}_\lambda(r,s) \equiv 0$. *Then*

$$\kappa(r,s) = \kappa(r,s-1) + \sigma(r,s-1) + \sum_{i=r}^{s-2} \kappa(r,i)\bar{\sigma}(i+1,s-1), \tag{95}$$

*if* $\lambda_s \not\equiv 0$.

*Moreover, if* $\lambda_s \equiv 0$ *then* $\kappa(r,s) = \kappa'(r,s-1)$.

*Remark.* Actually one can prove that the equality (95) holds in the case $\lambda_s \equiv 0$, too. However we do not need to prove it here since it will follow from Theorem 10.6 and Proposition 6.9.

*Proof.* By Corollary 10.4, we have

$$\kappa(r,s) = 1 - \nu(r,s) + \sum_{i=r}^{s-1} z_{i+1}\kappa'(r,i) - \sum_{i=r}^{s-1} \bar{\nu}(i+1,s)\kappa(r,i),$$

$$\kappa(r,s-1) = 1 - \nu(r,s-1) + \sum_{i=r}^{s-2} z_{i+1}\kappa'(r,i) - \sum_{i=r}^{s-2} \bar{\nu}(i+1,s-1)\kappa(r,i).$$

So

$$\begin{aligned} \kappa(r,s) - \kappa(r,s-1) = &-\big(\nu(r,s) - \nu(r,s-1)\big) + z_s\kappa'(r,s-1) \\ &- \sum_{i=r}^{s-2} \big(\bar{\nu}(i+1,s) - \bar{\nu}(i+1,s-1)\big)\kappa(r,i) \\ &- \bar{\nu}(s,s)\kappa(r,s-1). \end{aligned} \tag{96}$$

Since $\lambda_s > 0$ we have

$$z_s = 1. \tag{97}$$

Moreover, $\bar{\nu}(s,s) \neq 0$ if and only if $\lambda_s + 1 \not\equiv 0$. But in this case

$$\mathcal{B}(r,s-1) = \mathcal{B}(r,s) - (\lambda_s + 1) \not\equiv 0.$$

So $\kappa(r,s-1) = 0$ in view of Lemma 10.2. Thus,

$$\bar{\nu}(s,s)\kappa(r,s-1) = 0. \tag{98}$$

By Lemma 3.5 we have

$$\begin{aligned} &\bar{\nu}(i+1,s) - \bar{\nu}(i+1,s-1) \\ &= \begin{cases} 1, & \text{if } \bar{\nu}(i+1,s-1) \neq 0,\ \lambda_s \not\equiv 0,\ \mathcal{C}_{\lambda(i+1)}(i+1,s) \equiv 0; \\ -1, & \text{if } \bar{\nu}(i+1,s-1) \neq 0,\ \lambda_s \not\equiv 0,\ \mathcal{B}_{\lambda(i+1)}(i+1,s) \equiv 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{99}$$

Since $\mathcal{B}(r,s) \equiv 0$, the same lemma implies

$$-\big(\nu(r,s) - \nu(r,s-1)\big) = \begin{cases} 1, & \text{if } \nu(r,s-1) \neq 0,\ \lambda_s \not\equiv 0; \\ 0, & \text{otherwise.} \end{cases} \tag{100}$$

Now assume $\lambda_s \equiv 0$. Then $\mathcal{B}(r, s-1) = \mathcal{B}(r, s) - (\lambda_s + 1) \equiv -1 \not\equiv 0$. Therefore $\kappa(r, s-1) = 0$ in view of Lemma 10.2. Now (96)-(100) imply $\kappa(r, s) = \kappa'(r, s-1)$.

Let $\lambda_s \not\equiv 0$. Then we have from (100)

$$-\big(\nu(r, s) - \nu(r, s-1)\big) = \sigma(r, s-1). \tag{101}$$

Moreover, $\kappa(r, i)\big(\bar{\nu}(i+1, s) - \bar{\nu}(i+1, s-1)\big) \neq 0$ implies $\kappa(r, i) \neq 0$, hence $\mathcal{B}_\lambda(r, i) \equiv 0$ by Lemma 10.2. Note that $\mathcal{B}_\lambda(r, s) = \mathcal{B}_\lambda(r, i) + \mathcal{B}_{\lambda(i+1)}(i+1, s)$. So

$$\kappa(r, i)\big(\bar{\nu}(i+1, s) - \bar{\nu}(i+1, s-1)\big) \neq 0$$

implies $\mathcal{B}_{\lambda(i+1)}(i+1, s) \equiv 0$. But for such $i$ we have, in view of (99),

$$-\big(\bar{\nu}(i+1, s) - \bar{\nu}(i+1, s-1)\big) = \bar{\sigma}(i+1, s-1).$$

Now (96)-(100) imply

$$\kappa(r, s) = \kappa(r, s-1) + \kappa'(r, s-1) + \sigma(r, s-1) + \sum_{i=r}^{s-2} \kappa(r, i)\bar{\sigma}(i+1, s-1).$$

It remains to note that $\mathcal{B}_{\lambda(s-1)}(r, s-1) = \mathcal{B}_\lambda(r, s) - \lambda_s \not\equiv 0$ so $\kappa'(r, s-1) = 0$ in view of Lemma 10.2. □

Recall the constants $q_{r,s}(\lambda)$ defined in 6.5.

**Theorem 10.6.** *Let $\lambda \in P_+(A_l)$, $1 \leq r \leq s \leq l$. Then*

$$[V(\lambda) : L(\lambda - \alpha_r - \alpha_{r+1} - \cdots - \alpha_s)] = q_{r,s}(\lambda).$$

*Proof.* We proceed by induction on $s - r$. If $s - r = 0$ then the result follows from 10.1 and 6.5. Let $s - r > 0$, and the theorem have been proved for all $r', s'$ with $s' - r' < s - r$. If $\mathcal{B}(r, s) \not\equiv 0$ or $\lambda_r = 0$ or $\lambda_s = 0$ then the result follows from 10.2 and 6.5. Assume $\mathcal{B}(r, s) \equiv 0$, $\lambda_r > 0$, $\lambda_s > 0$. Then Corollary 10.5 yields

$$\kappa(r, s) = \kappa(r, s-1) + \sigma(r, s-1) + \sum_{i=r}^{s-2} \kappa(r, i)\bar{\sigma}(i+1, s-1), \text{ if } \lambda_s \not\equiv 0,$$
$$\kappa(r, s) = \kappa'(r, s-1), \text{ if } \lambda_s \equiv 0.$$

On the other hand,

$$q_{r,s} = q_{r,s-1} + \sigma(r, s-1) + \sum_{i=r}^{s-2} q(r, i)\bar{\sigma}(i+1, s-1), \text{ if } \lambda_s \not\equiv 0,$$
$$q_{r,s} = q'_{r,s-1}, \text{ if } \lambda_s \equiv 0$$

by Proposition 6.9. Now it suffices to apply the inductive hypothesis. □

## 11. How to translate from $SL_n$ to $\Sigma_n$

Throughout the section $n = l + 1$. We denote by $\nu'$ the partition conjugate to a partition $\nu$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P_+(A_l)$ be a dominant weight. Assume that

$$\sum_{i=1}^{l} i\lambda_i = n.$$

Then we can define partitions of $n$:

$$\varphi_n(\lambda) = (\lambda_1 + \lambda_2 + \cdots + \lambda_l, \lambda_2 + \cdots + \lambda_l, \dots, \lambda_l),$$
$$\psi_n(\lambda) = \varphi_n(\lambda)'.$$

We also define

$$\psi_n(0) = (n).$$

On the other hand, let us represent a partition $\mu$ of $n$ in the form

$$\mu = (\mu_1^{b_1}, \mu_2^{b_2}, \dots, \mu_k^{b_k}); \quad \mu_1 > \mu_2 > \cdots > \mu_k; \quad b_1, b_2, \dots, b_k > 0.$$

If $\mu \neq (n)$ we define a dominant weight $w_n(\mu) \in P_+(A_l)$ by setting

$$w_n(\mu) = b_k\omega_{\mu_k} + b_{k-1}\omega_{\mu_{k-1}} + \cdots + b_1\omega_{\mu_1}.$$

We also set

$$w_n((n)) = 0 \in P_+(A_l).$$

One clearly has

$$w_n \circ \psi_n = \mathrm{id}, \quad \psi_n \circ w_n = \mathrm{id}\,.$$

The following fundamental result identifies the decomposition numbers of $\Sigma_n$ with certain decomposition numbers of $SL_n$ (the result follows immediately from the corresponding result for $GL_n$ proved in [6]).

**Proposition 11.1.** [6] *Let $\mu, \nu$ be partitions of $n$ and $\nu$ be $p$-regular. Then*

$$[S^\mu : D^\nu] = [V(w_n(\mu)) : L(w_n(\nu))]\,.$$

The question on branching multiplicities for $\Sigma_n$ can also be reduced to a similar question for $SL_n$ (we assumed $\nu \neq (n)$ in [13] but for $\nu = (n)$ the statement is trivial).

**Proposition 11.2.** [13, 2.12(v)] *Let $\mu$ be a $p$-regular partition of $n$ and $\nu$ be a $p$-regular partition of $n-1$. Then*

$$[D^\mu \downarrow_{\Sigma_{n-1}} : D^\nu] = \left[L(w_n(\mu))_1 \mid A_{l-1}(K) : L(w_{n-1}(\nu))\right].$$

Finally we can identify the multiplicities of the Young modules $Y^\nu$ in the permutation modules $M^\mu$ (over $\Sigma_n$) with weight multiplicities of the irreducible modules $L(w_n(\mu'))$ (over $SL_n$). The next result is due to Klyachko.

**Proposition 11.3.** [17] (*see also* [18]) *Let $\lambda$ and $\mu$ be partitions of $n$. Then*

$$[M^\lambda : Y^\mu] = \dim L(w_n(\mu'))_{w_n(\lambda')}.$$

We now explain how Theorem 1.4 can be deduced from Theorem 9.3 and Proposition 11.2.

Let $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ be a $p$-regular partition of $n$. We may assume $\lambda \neq (n)$ since then Theorem 1.4 is evident. Recall that

$$\lambda(j) = (l_1, \dots, l_{j-1}, l_j - 1, l_{j+1}, \dots, l_m)$$

(see (2)). Then

$$w_{n-1}(\lambda(j)) = \overline{w_n(\lambda) - \alpha(l_j, l)}.$$

So in view of Proposition 11.2,

$$\left[D^\lambda \downarrow_{\Sigma_{n-1}} : D^{\lambda(j)}\right] = \left[L(w_n(\lambda))_1 \mid A_{l-1}(K) : L(\overline{w_n(\lambda) - \alpha(l_j, l)})\right].$$

However, the multiplicities $\left[L(w_n(\lambda))_1 \mid A_{l-1}(K) : L(\overline{w_n(\lambda) - \alpha(l_j, l)})\right]$ have been calculated in Theorem 9.3. Now to deduce Theorem 1.4 we need only to note that

$$\begin{aligned} &l_k - l_j + j - k + 1 = \mathcal{B}_{w_n(\lambda)}(l_j, l_k) \\ &\text{for any } 1 \leq k < j \leq m \text{ with } l_k < l_{k-1} \text{ and } l_j > l_{j+1}, \end{aligned} \tag{102}$$

and

$$\begin{aligned} &l_k - l_j + j - k = \mathcal{C}_{w_n(\lambda)}(l_j, l_k) \\ &\text{for any } 1 \leq k < j \leq m \text{ with } l_k > l_{k+1} \text{ and } l_j > l_{j+1}. \end{aligned} \tag{103}$$

Moreover, note that if $\lambda \neq (n)$ then

$$w_n(\lambda(i,j)) = w_n(\lambda) - \alpha(l_j, l_i).$$

So to deduce Theorem 1.10 it suffices to use Theorem 10.6, Proposition 11.1 and (102), (103).

Besides,

$$w_n(\lambda') = w_n(\lambda(i,j)') - \alpha(i, j-1).$$

So, by Proposition 11.3, the description of the multiplicities $[M^\lambda : Y^{\lambda(i,j)}]$ follows immediately from Corollary 8.10.

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Department of Mathematics, University of Oregon, Eugene OR 97403, USA
*E-mail address*: klesh@math.uoregon.edu