BRANCHING RULES FOR MODULAR REPRESENTATIONS OF SYMMETRIC GROUPS, IV

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Introduction

Let Σ_n be the symmetric group on n letters, and D be an irreducible representation of Σ_n over a ground field F of charactersitic $p > 0$. The main results of this paper are:

(1) We describe the algebra $\mathrm{End}_{\Sigma_{n-1}}(D\downarrow_{\Sigma_{n-1}})$;

(2) we obtain a criterion for the semisimplicity of the induced module $D\uparrow^{\Sigma_{n+1}}$, and in the case of semisimplicity describe its composition factors.

We note that for characteristic $p = 2$ the criterion for the semisimplicity we get was conjectured by Benson [1, Conjecture 5]. Thus we generalize to an arbitrary p and prove the last unproved conjecture made by Benson at the Arcata Conference. Actually, (2) easily follows from (1) using a Mackey type theorem together with our previous results on $D\uparrow^{\Sigma_{n+1}}$ [15].

We recall the necessary definitions and give the exact formulations of the main results. The list of the notions defined in [15] and used here is as follows.

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ is a (fixed) ***p-regular partition*** of n;

$\lambda = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid j \leq \lambda_i\}$ is its ***Young diagram*** (we do not distinguish between partitions and their Young diagrams);

D^λ is the irreducible $F\Sigma_n$-module corresponding to λ [9];

$(i,j) \in \mathbb{N}\times\mathbb{N}$ is called a ***node***;

$(i,\lambda_i) \in \lambda$ is called a ***removable node*** (of λ) if $\lambda_i > \lambda_{i+1}$;

(i,λ_i+1) is called an ***addable node*** (for λ) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$;

$\lambda_A = \lambda \setminus \{A\} = (\lambda_1,\dots,\lambda_{i-1},\lambda_i - 1,\lambda_{i+1},\dots)$ is a partition of $n-1$ obtained by removing a removable node $A = (i,\lambda_i)$ from λ;

$\lambda^B = \lambda \cup \{B\} = (\lambda_1,\dots,\lambda_{i-1},\lambda_i + 1,\lambda_{i+1},\dots)$ is a partition of $n+1$ obtained by adding an addable node $B = (i,\lambda_i+1)$ to λ;

$\mathrm{res}\, A = j - i \pmod p$ is the (p)-***residue*** of a node $A = (i,j)$.

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A removable node A of λ is called ***normal*** if for every addable node B above A with $\mathrm{res}\, B = \mathrm{res}\, A$ there exists a removable node $C(B)$ strictly between A and B with $\mathrm{res}\, C(B) = \mathrm{res}\, A$, and $B \neq B'$ implies $C(B) \neq C(B')$. A removable node is called ***good*** if it is the lowest among the normal nodes of a fixed residue. An addable node B is called ***good*** if it is good as a removable node of λ^B.

The first main result of the paper is

Theorem A.

(i) *The dimension of* $\mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}})$ *is equal to the number of normal nodes;*

(ii) *the dimension of* $\mathrm{End}_{\Sigma_{n+1}}(D^\lambda \uparrow^{\Sigma_{n+1}})$ *is one more than the number of normal nodes.*

Actually we can describe the algebra $\mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}})$ explicitly. Assume that the normal nodes of λ have residues $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{Z}/p\mathbb{Z}$ (there may be several or no normal nodes of a fixed residue), and let $A_1, A_2, \dots, A_s$ be the good nodes with the corresponding residues. It is proved in [15] that

$$D^\lambda \downarrow_{\Sigma_{n-1}} \cong I_1 \oplus I_2 \oplus \dots \oplus I_s$$

where $I_1, I_2, \dots I_s$ are indecomposable, belong to distinct blocks, and the socle of I_l is $D^{\lambda_{A_l}}$, $l = 1, 2, \dots, s$. Hence

$$\mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}}) \cong \mathrm{End}_{\Sigma_{n-1}}(I_1) \oplus \mathrm{End}_{\Sigma_{n-1}}(I_2) \oplus \dots \oplus \mathrm{End}_{\Sigma_{n-1}}(I_s).$$

Now the algebras $\mathrm{End}_{\Sigma_{n-1}}(I_l)$, $l = 1, 2, \dots, s$, are described in the following

Theorem B. *Let $1 \le l \le s$. Suppose there are $m(l)$ normal nodes of residue α_l. There exists an endomorphism $\varphi_l \in \mathrm{End}_{\Sigma_{n-1}}(I_l)$ such that $\varphi_l^{m(l)} = 0$ and*

$$\{\mathrm{id}_{I_l}, \varphi_l, \varphi_l^2, \dots, \varphi_l^{m(l)-1}\}$$

is a linear basis of $\mathrm{End}_{\Sigma_{n-1}}(I_l)$. In particular, $\mathrm{End}_{\Sigma_{n-1}}(I_l)$ is isomorphic to the algebra of truncated polynomials $F[x]/(x^{m(l)})$.

Now we turn to the criterion for $D^\lambda \uparrow^{\Sigma_{n+1}}$ to be semisimple.

Theorem C. *Let λ have r good addable nodes and s normal removable nodes. Then $D^\lambda \uparrow^{\Sigma_{n+1}}$ is semisimple if and only if $r = s + 1$, in which case*

$$D^\lambda \uparrow^{\Sigma_{n+1}} \cong \oplus D^{\lambda^B},$$

the sum being taken over all good addable nodes B.

If $p = 2$, the condition in Theorem C is equivalent to the one conjectured by Benson [1, Conjecture 5]:

Corollary D. *Let $p = 2$ and $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_t > 0)$ be a 2-regular partition. Then $D^\lambda \uparrow^{\Sigma_{n+1}}$ is semisimple if and only if all λ_i are even. In this case*

$$D^\lambda \uparrow^{\Sigma_{n+1}} \cong D^{(\lambda_1, \dots, \lambda_{t-1}, \lambda_t + 1)} \oplus D^{(\lambda_1, \dots, \lambda_t, 1)}.$$

As a by-product of our calculations with $GL(n)$ (in Section 2 of this paper), we get a curious filtration of the indecomposables I_l. Let S^λ be a ***Specht module*** (see [9]). A quotient module T of S^λ is called a ***small quotient of*** S^λ if the following two conditions hold:

(1) T has no composition factors D^μ with μ obtained from λ by moving one box up;

(2) if λ is p-regular then $T \neq 0$.

As before Theorem B, we fix a residue α_l, and let $C(1), C(2), \dots, C(m(l))$ be all the normal nodes of residue α_l listed from bottom to top ($1 \le l \le s$). Consider the set of pairs

$$\{(j,k) \mid 1 \le j \le k \le m(l)\}$$

totally ordered as follows:

$$(j,k) \prec (t,u) \;\Leftrightarrow\; 2k - j < 2u - t \text{ or } 2k - j = 2u - t \text{ and } j < t.$$

We denote by $(j,k)'$ the pair immediately preceding (j,k) in this total order. (Of course $(1,1)'$ is not defined—below we interpret $\Psi_{(1,1)'}$ as 0.)

Theorem E. *Let $1 \le l \le s$. There exists a filtration $\{\Psi_{(j,k)} \mid 1 \le j \le k \le m(l)\}$ of I_l with $\Psi_{(j,k)} \subseteq \Psi_{(t,u)}$ if and only if $(j,k) \prec (t,u)$ such that*

(1) $\Psi_{(1,m(l))} = I_l$;

(2) every consecutive quotient $F_{(j,k)} = \Psi_{(j,k)}/\Psi_{(j,k)'}$ is a small quotient of $S^{\lambda_{C(j)}}$;

(3) Let $k_2 > k_1 \ge j$. Then $F_{(j,k_1)}$ is a quotient of $F_{(j,k_2)}$. Moreover,

$$\varphi_l^{k_2-k_1}(\Psi_{(j,k_2)}) \subseteq \Psi_{(j,k_1)}, \quad \varphi_l^{k_2-k_1}(\Psi_{(j,k_2)'}) \subseteq \Psi_{(j,k_1)'},$$

and in this way $\varphi_l^{k_2-k_1}$ induces a surjective homomorphism from $F_{(j,k_2)}$ to $F_{(j,k_1)}$.

Example. Let $m(l) = 3$. Then we have a filtration

$$(0) \subseteq \Psi_{(1,1)} \subseteq \Psi_{(2,2)} \subseteq \Psi_{(1,2)} \subseteq \Psi_{(3,3)} \subseteq \Psi_{(2,3)} \subseteq \Psi_{(1,3)} = I_l,$$

and $\Psi_{(1,1)}$, $\Psi_{(2,2)}/\Psi_{(1,1)}$, $\Psi_{(1,2)}/\Psi_{(2,2)}$, $\Psi_{(3,3)}/\Psi_{(1,2)}$, $\Psi_{(2,3)}/\Psi_{(3,3)}$, $\Psi_{(1,3)}/\Psi_{(2,3)}$ are small quotients of $S^{\lambda_{C(1)}}$, $S^{\lambda_{C(2)}}$, $S^{\lambda_{C(1)}}$, $S^{\lambda_{C(3)}}$, $S^{\lambda_{C(2)}}$, $S^{\lambda_{C(1)}}$, respectively. Moreover, $\Psi_{(1,1)}$ is a quotient of $\Psi_{(1,2)}/\Psi_{(2,2)}$, which is in turn a quotient of $\Psi_{(1,3)}/\Psi_{(2,3)}$, and $\Psi_{(2,2)}/\Psi_{(1,1)}$ is a quotient of $\Psi_{(2,3)}/\Psi_{(3,3)}$. (Actually, it is known that $\Psi_{(1,1)}$ is always equal to $D^{\lambda_{C(1)}}$).

Remark. In general, $\prec$ is not the only total order on $\{(j,k) \mid 1 \le j \le k \le m(l)\}$ for which Theorem E holds.

The following result, first proved in [16], is a special case of Theorem E (but it was used in the proof of this theorem).

Corollary F. [16, 1.4] *Let A be a removable node of λ, and λ_A be p-regular. Then the multiplicity $[D^\lambda\downarrow_{\Sigma_{n-1}} : D^{\lambda_A}]$ is non-zero if and only if A is normal. Moreover,* *for A normal we have*

$$[D^\lambda\downarrow_{\Sigma_{n-1}}: D^{\lambda_A}] = 1 + \#\{\text{normal } B \text{ above } A \text{ with } \operatorname{res} B = \operatorname{res} A\}$$

Using the now standard technique, we translate our problems from Σ_n to $GL(n) = GL_n(F)$ (we assume F to be algebraically closed and all modules to be rational when we work with the algebraic group $GL(n)$). Then we apply the lowering operators introduced in [14] to do with the arising problems for $GL(n)$. We use various properties of these operators proved in [14] and [2]. These properties are gathered in Section 1. In the same section we also deal with Murphy operators introduced in [2]. They are closely related with a "Casimir operator" which ultimately provides us with the endomorphisms we are looking for. The main results about $GL(n)$ are proved in Section 2. They are translated into "Σ_n-language" in Section 3. The Benson's Conjecture is proved in Section 4.

We complete this introduction with the following conjecture generalizing one of the main results of the paper.

Conjecture. Denote by $L_n(\lambda)$ (resp., $\Delta_n(\lambda)$) the irreducible (resp., Weyl) module with high weight λ. Following [2], define a dominant $GL(n-1)$-weight μ to be λ-normal if

$$\operatorname{Hom}_{GL(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)\downarrow_{GL(n-1)}) \neq 0.$$

We conjecture that $\dim \operatorname{End}_{GL(n-1)}(L_n(\lambda)\downarrow_{GL(n-1)})$ is equal to the number of λ-normal weights.

1. Preliminaries on $GL(n)$

The category of finite dimensional rational $FGL(n)$-modules is equivalent to the category of finite dimensional modules over the hyperalgebra $U(n)$ of $GL(n)$. We are going to work in the latter one. The main reference here is [17].

To define $U(n)$ we first consider the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ with the basis consisting of the matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. The elements

$$\{\frac{e_{ij}^k}{k!}, \begin{pmatrix} e_{ii} \\ k \end{pmatrix} \mid 1 \leq i \neq j \leq n,\ k \geq 0\}$$

generate a $\mathbb{Z}$-subalgebra $U(n, \mathbb{Z})$ in the universal enveloping $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$. Now $U(n)$ is $U(n, \mathbb{Z}) \otimes_{\mathbb{Z}} F$. We use special names for certain elements of $U(n)$:

$$E_{ij}^{(k)} = \frac{e_{ij}^k}{k!} \otimes 1, \quad F_{ij}^{(k)} = \frac{e_{ji}^k}{k!} \otimes 1, \quad 1 \leq i < j \leq n;$$

$$\begin{pmatrix} H_i \\ k \end{pmatrix} = \begin{pmatrix} e_{ii} \\ k \end{pmatrix} \otimes 1,\ 1 \leq i \leq n.$$

We also write E_{ij} for $E_{ij}^{(1)}$, H_i for $\begin{pmatrix} H_i \\ 1 \end{pmatrix}$, F_{ij} for $F_{ij}^{(1)}$. The subalgebras of $U(n)$ generated by

$$\{E_{ij}^{(k)} \mid 1 \leq i < j \leq n,\ k \geq 0\},$$

$$\{\binom{H_i}{k} \mid 1 \leq i \leq n,\ k \geq 0\},$$

$$\{F_{ij}^{(k)} \mid 1 \leq i < j \leq n,\ k \geq 0\}$$

are denoted by $U(n)^+$, $U(n)^0$, $U(n)^-$, respectively. By a version of PBW-theorem, proved in [17, Theorem 2],

(1) $\{F_{1,2}^{(N_{1,2})} F_{1,3}^{(N_{1,3})} F_{2,3}^{(N_{2,3})} \dots F_{1,n}^{(N_{1,n})} F_{2,n}^{(N_{2,n})} \dots F_{n-1,n}^{(N_{n-1,n})} \mid N_{ij} \geq 0\}$ is a basis of $U(n)^-$.

There are natural embeddings of $U(n-1), U(n-1)^-$, etc., into $U(n)$ corresponding to the embedding of $GL(n-1)$ into the top left hand corner of $GL(n)$.

If $\lambda = (\lambda_1, \dots, \lambda_n)$ is an n-tuple of integers, we can identify it with the ***weight***

$$\lambda : U(n)^0 \to F,\ \binom{H_i}{k} \mapsto \binom{\lambda_i}{k}.$$

A weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is called ***dominant*** if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The set of all weights is denoted by $X(n)$ and the set of all dominant weights is denoted by $X_+(n)$.

If $\lambda, \mu \in X(n)$ we say λ ***dominates*** μ, written $\lambda \geq \mu$, if $\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_i$ for all $j = 1, 2, \dots, n$.

It is convenient to use the weights

$$\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0),$$

with 1 on the ith position, $i = 1, 2, \dots, n$. Then

$$\alpha(i,j) := \varepsilon_i - \varepsilon_j,\ 1 \leq i < j \leq n,$$

are the ***positive roots***, and

$$\alpha(i, i+1),\ i = 1, 2, \dots, n-1,$$

are the ***simple roots***.

If μ is a weight we denote by $U(n)^-_{-\mu}$ the vector subspace of $U(n)^-$ spanned by all monomials of the basis in (1) such that

$$N_{1,2}\alpha(1,2) + N_{1,3}\alpha(1,3) + \dots + N_{n-1,n}\alpha(n-1,n) = \mu.$$

$U(n)^+_\mu$ is defined similarly.

Finally, for $\lambda = (\lambda_1, \dots, \lambda_n) \in X(n)$ we denote by $\lambda \downarrow$ the weight $(\lambda_1, \dots, \lambda_{n-1}) \in X(n-1)$.

We use $[i,j]$, $(i,j]$, etc., to denote the intervals of *integers*, for example $[i,j) = \{k \in \mathbb{Z} \mid i \leq k < j\}$.

We write $a \equiv b$ for $a \equiv b \pmod p$.

From now on we fix a dominant weight $\lambda \in X_+(n)$. For $1 \leq i \leq j < n$ we define

$$B(i,j) = \lambda_i - \lambda_{j+1} + j - i,$$
$$C(i,j) = \lambda_i - \lambda_j + j - i.$$

For $1 \leq i \leq j \leq n$ put

$$\mathcal{B}(i,j) = \{k \in [i,j) \mid B(i,k) \equiv 0\},$$
$$\mathcal{C}(i,j) = \{k \in (i,j) \mid C(i,k) \equiv 0\}.$$

For two sets $\mathcal{C}$ and $\mathcal{B}$ of integers, a ***weakly decreasing*** (resp., ***increasing***) ***injection*** from $\mathcal{B}$ to $\mathcal{C}$ is an injective map $\varphi : \mathcal{B} \to \mathcal{C}$ such that $\varphi(b) \leq b$ (resp., $\varphi(b) \geq b$) for any $b \in \mathcal{B}$. We say $\mathcal{C}$ ***compensates*** $\mathcal{B}$ if there exists a weakly decreasing injection from $\mathcal{B}$ to $\mathcal{C}$.

Let $1 \leq i < j \leq n$. We say i is ***normal in*** $[i,j)$ if $\mathcal{C}(i,j)$ compensates $\mathcal{B}(i,j)$. We say i is ***normal*** if it is normal in $[i,n)$. It follows from the definitions that

$$\text{if } i \text{ is normal then } \lambda_i - \lambda_{i+1} \not\equiv 0. \tag{2}$$

The following combinatorial lemma was proved in [16, 3.4].

Lemma 1.1. *Let i be normal in $[i,j)$, and $|\mathcal{B}(i,j)| < |\mathcal{C}(i,j)|$. Then there exists $c \in \mathcal{C}(i,j)$ which is normal in $[c,j)$.*

We denote by $L_n(\lambda)$ and $\Delta_n(\lambda)$ the irreducible and the Weyl module, respectively, over $U(n)$ with ***high weight*** λ. Let v_+ be a ***high weight vector*** of $L_n(\lambda)$.

Let w be a weight vector of a $U(n)$-module V. Then the vector w is called $(U(n)\text{-})$***primitive*** if $E_{ij}^{(k)} w = 0$ for all $1 \leq i < j \leq n$, $k > 0$. It is called $U(n-1)$-***primitive*** if $E_{ij}^{(k)} w = 0$ for all $1 \leq i < j \leq n-1$, $k > 0$. Thus, a weight vector is $U(n-1)$-primitive if it is a "high weight vector" for $U(n-1)$ as embedded earlier.

If V is a $U(n)$-module and μ is a weight we denote by V_μ the corresponding ***weight space*** of V.

We now recall some properties of ***lowering operators*** proved in [14]. We note that Brundan [3, 4] found easier proofs of these properties. Observe that our $T_{ij}(M)$ corresponds to $T_{i,j-1}(M)$ from [14] and to $S_{ij}((i,j) \setminus M)$ from [3, 4, 2]. (Actually our $T_{ij}(M)$ is obtained by evaluating Brundan's $S_{ij}((i,j) \setminus M)$ at weight λ.)

Let $1 \leq i < j \leq n$. For $U = \{u_1 < u_2 < \dots < u_a\} \subseteq (i,j)$ we set

$$F_{ij}(U) = F_{i,u_1} F_{u_1,u_2} \dots F_{u_a,j} \in U(n)^- \quad (F_{ij}(\emptyset) = F_{ij}).$$

By (1), $\{F_{ij}(U) \mid U \subseteq (i,j)\}$ is a basis of $U(n)^-_{-\alpha(i,j)}$. It is shown in [14] that there exists another basis

$$\{T_{ij}(M) \mid M \subseteq (i,j)\}$$

of $U(n)^-_{-\alpha(i,j)}$ with certain remarkable properties. Some of these properties are listed below for future reference.

Lemma 1.2. [14, 2.17]

$$T_{ij}(M) = \Big(\prod_{k \in (i,j) \setminus M} C(i,k) \Big) F_{ij} + [\textit{a linear combination of } F_{ij}(U) \textit{ with } U \neq \emptyset].$$

Lemma 1.3. [14, 2.14] *Let* $1 \le i < k < j \le n$, $M \subseteq (i,k)$, $N \subseteq (k,j)$, $m = \max M$ *interpreted as* i *if* $M = \emptyset$. *Then*

$$\begin{aligned} T_{ik}(M)T_{kj}(N) = & T_{ij}(M \cup N) - T_{ij}(M \setminus \{m\} \cup \{k\} \cup N) \\ & - C(m,k)T_{ij}(M \cup \{k\} \cup N), \end{aligned}$$

with the second term omitted if $M = \emptyset$.

The following result tells how the E's corresponding to all but one simple roots act on the lowering operators.

Lemma 1.4. [14, 2.13] *Let* $k \in [1,n) \setminus \{j-1\}$. *Then*

$$\begin{aligned} & E_{k,k+1}T_{ij}(M)v_+ \\ & = \begin{cases} -T_{ik}(M \cap (i,k))T_{k+1,j}(M \cap (k+1,j))v_+ & \textit{if } k \notin M \textit{ but } k+1 \in M; \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

The next lemma provides a criterion for $T_{ij}(M)v_+$ to be zero.

Lemma 1.5. [14, 3.2] $T_{ij}(M)v_+ = 0$ *if and only if there exists* $k \in (i,j] \setminus M$ *and a weakly increasing injection* $\varphi : \{i\} \cup (M \cap (i,k)) \to [i,k)$ *such that* $B(m, \varphi(m)) \equiv 0$ *for any* $m \in \{i\} \cup (M \cap (i,k))$.

Definition 1.6. Let $1 \le i < j \le n$. We say that $F_{ij}v_+$ ***can be expressed through non-trivial subdivisions***, if $F_{ij}v_+$ is a linear combination of $F_{ij}(U)v_+$ with $U \ne \emptyset$ (in particular, if $F_{ij}v_+ = 0$).

The following result tells us exactly when $F_{ij}v_+$ can be expressed through non-trivial subdivisions (for a generalization see [2, 5.2], [5, 3.3]).

Lemma 1.7. [14, 4.6,4.2] $F_{ij}v_+$ *can be expressed through non-trivial subdivisions if and only if* i *is not normal in* $[i,j)$.

We recall the definition and some properties of the Murphy operators introduced in [2]. Set

$$L_j = \sum_{i=1}^{j-1} F_{ij}E_{ij}, \ j = 2,3,\ldots,n. \tag{3}$$

These are ***Murphy operators***. It is easy to check that they commute. A remarkable fact first observed by Brundan is that Murphy operators can be used to define lowering operators $T_{ij}(M)$:

Lemma 1.8. [2, Section 3.3]

$$T_{ij}(M)v_+ = \prod_{k \in (i,j) \setminus M} (C(i,k) - L_{i+1} - L_{i+2} - \cdots - L_k)F_{ij}v_+$$

(one does not need to specify the order in the product since the factors commute).

Another result proved in [2] is useful in calculations:

Lemma 1.9. [2, 3.2(ii)] *Let $i < k < j$. Then*

$$(C(i,k) - L_{i+1} - L_{i+2} - \dots - L_k)L_k F_{ij} v_+ = 0.$$

We prove a couple of further results concerning the L_k.

Lemma 1.10. *Assume $1 \le i < j < k \le n$. Then $L_k F_{ij} = F_{ij} L_k$.*

Proof. We have

$$\begin{aligned} L_k F_{ij} &= \sum_{l=1}^{k-1} F_{lk} E_{lk} F_{ij} = \sum_{l=1}^{k-1} F_{lk} F_{ij} E_{lk} + F_{ik}[E_{ik}, F_{ij}] \\ &= \sum_{l=1}^{k-1} F_{ij} F_{lk} E_{lk} + [F_{jk}, F_{ij}]E_{jk} + F_{ik}[E_{ik}, F_{ij}]. \end{aligned}$$

It remains to use $[F_{jk}, F_{ij}] = F_{ik}$, $[E_{ik}, F_{ij}] = -E_{jk}$. □

Lemma 1.11. *Let $k \in (1, n] \setminus \{j\}$. Then*

$$L_k T_{ij}(M) v_+ = \begin{cases} -T_{ik}(M \cap (i,k)) T_{kj}(M \cap (k,j)) v_+ & \textit{if } k \in M; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $k \le i$ or $k > j$ then any E_{lk} annihilates $T_{ij}(M)v_+$ since the weight of $E_{lk} T_{ij}(M) v_+$ is $\lambda - \alpha(i,j) + \alpha(l,k)$ which is not dominated by λ in this case. So we may assume that $k \in (i,j)$. If $k \in (i,j) \setminus M$ then $L_k T_{ij}(M) v_+ = 0$ by Lemmas 1.8 and 1.9.

Finally, assume $k \in M$. Let $m = \max(M \cap (i,k))$ interpreted as i if $M \cap (i,k) = \emptyset$. Then, again using Lemmas 1.8, 1.9 and what have already been proved, we get

$$\begin{aligned} & L_k T_{ij}(M) v_+ \\ & = -(C(i,k) - L_{i+1} - \dots - L_k) T_{ij}(M) v_+ \\ & \quad + (C(i,k) - L_{i+1} - \dots - L_{k-1}) T_{ij}(M) v_+ \\ & = -T_{ij}(M \setminus \{k\}) v_+ \\ & \quad + (C(i,m) - L_{i+1} - \dots - L_m + C(m,k) - L_{m+1} - \dots - L_{k-1}) T_{ij}(M) v_+ \\ & = -T_{ij}(M \setminus \{k\}) v_+ + T_{ij}(M \setminus \{m\}) v_+ + C(m,k) T_{ij}(M) v_+, \end{aligned}$$

with the second term omitted if $M \cap (i,k) = \emptyset$. But the last expression equals

$$-T_{ik}(M \cap (i,k)) T_{kj}(M \cap (k,j)) v_+$$

by virtue of Lemma 1.3. □

We now need a "Casimir element" $z_{n-1} \in U(n-1)$ which is the Carter-Lusztig's central element C_2, see [6, 2.2]. Set

$$z_{n-1} := \sum_{1 \le i < j \le n-1} ((H_i - i)(H_j - j) - F_{ij} E_{ij}) = \theta_{n-1} - L_2 - L_3 - \dots - L_{n-1}, \tag{4}$$

where

$$\theta_{n-1} = \sum_{1\le i<j\le n-1} (H_i - i)(H_j - j) \in U(n-1)^0 \subset U(n)^0.$$

It follows from [6, Corollary 2.2] that

$$z_{n-1} \text{ is central in } U(n-1). \tag{5}$$

Lemma 1.12. *Assume* $1 \le i_1 < i_2 < n$. *Then*

$$(\lambda - \alpha(i_1, n))(\theta_{n-1}) - (\lambda - \alpha(i_2, n))(\theta_{n-1}) = C(i_1, i_2). \tag{6}$$

Proof. We have for $l = 1, 2$:

$$\lambda - \alpha(i_l, n) = (\lambda_1, \dots, \lambda_{i_l-1}, \lambda_{i_l} - 1, \lambda_{i_l+1}, \dots, \lambda_{n-1}, \lambda_n + 1).$$

So

$$(\lambda - \alpha(i_l, n))(\theta_{n-1}) = \sum_{\substack{1\le j<k\le n-1 \\ j,k\ne i_l}} (\lambda_j - j)(\lambda_k - k) + \sum_{j=1}^{i_l-1} (\lambda_j - j)(\lambda_{i_l} - 1 - i_l)$$
$$+ \sum_{k=i_l+1}^{n-1} (\lambda_{i_l} - 1 - i_l)(\lambda_k - k).$$

Now a simple calculation proves (6). □

2. A filtration of the first level

The representation

$$\det{}^c : GL(n) \to GL(1),\ g \mapsto (\det g)^c$$

has high weight $(c, c, \dots, c)$, $c \in \mathbb{Z}$. It follows that

$$L_n(\lambda_1, \lambda_2, \dots, \lambda_n) \otimes \det{}^c \cong L_n(\lambda_1 + c, \lambda_2 + c, \dots, \lambda_n + c).$$

Since all interesting properties of $L_n(\lambda)$ are invariant with respect to such an operation we may assume that $\lambda_n = 0$. Thus, ***from now on***,

$$\lambda \text{ is an arbitrary dominant weight of the form}$$
$$\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n), \text{ with } \lambda_n = 0$$

This is convenient because now we can consider the ***Young diagram*** of λ. If (i, j) is a node of λ we define its p-***residue*** to be $(i - j) \pmod p$. (Observe that when we work with the symmetric group the residue is defined to be $(j - i) \pmod p$. The reason for the difference comes from the fact that Schur functors transpose diagrams). A node (i, λ_i) is called ***normal*** if and only if i is normal as defined in Section 1. It follows from (2) that normal nodes are removable. (Note again that the normality just defined is equivalent to the normality in the introduction but up to transposing Young diagrams.)

Fix a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let

$$\Gamma_\alpha = \{i_1^{(\alpha)} < i_2^{(\alpha)} < \dots < i_{m_\alpha}^{(\alpha)}\} = \{i \mid i \text{ is normal and } \operatorname{res}(i, \lambda_i) = \alpha\}. \tag{7}$$

Since α is fixed we often omit α everywhere and write $\Gamma = \{i_1 < i_2 < \cdots < i_m\}$. Of course Γ might be empty, which will turn out to be an easy case. So we now concentrate on the case where $\Gamma \neq \emptyset$. Note that $C(i,j) = \lambda_i - \lambda_j + j - i$ is just the difference between the residues of (j, λ_j) and (i, λ_i) so

$$C(i_j, i_k) \equiv 0 \text{ for all } 1 \leq j \leq k \leq m,$$

and

$$C(\min(i_j, l), \max(i_j, l)) \not\equiv 0$$

for any normal $l \notin \{i_1, i_2, \dots, i_m\}$ and any $j \in \{1, 2, \dots, m\}$. Set

$$\mathcal{B}_j = \mathcal{B}(i_j, i_{j+1}), \quad \mathcal{C}_j = \mathcal{C}(i_j, i_{j+1}), \quad \bar{\mathcal{C}}_j = \{i_j\} \sqcup \mathcal{C}_j, \quad j = 1, 2, \dots, m.$$

(we always interpret i_{m+1} as n). Also put

$$\mathcal{B}_{jk} = \cup_{j \leq l \leq k} \mathcal{B}_l, \quad C_{jk} = \cup_{j \leq l \leq k} \mathcal{C}_l, \quad \bar{\mathcal{C}}_{jk} = \cup_{j \leq l \leq k} \bar{\mathcal{C}}_l, \quad 1 \leq j, k \leq m.$$

Note that

$$B(i_j, i_{j+1} - 1) = C(i_j, i_{j+1}) - 1 \equiv -1 \not\equiv 0, \; j = 1, 2, \dots, m-1.$$

This together with

$$B(i_j, i_j) = \lambda_{i_j} - \lambda_{i_j+1} \not\equiv 0$$

(see (2)) implies

$$\mathcal{B}_j \subseteq (i_j, i_{j+1} - 2], \quad j = 1, 2, \dots, m-1. \tag{8}$$

Lemma 2.1. *Let $1 \leq j \leq m$. Then*

(i) $|\mathcal{C}_j| = |\mathcal{B}_j|$;
(ii) $\mathcal{C}_j$ *compensates* $\mathcal{B}_j$;
(iii) $\mathcal{C}(i_j, n) = \mathcal{C}_j \sqcup \bar{\mathcal{C}}_{j+1} \sqcup \cdots \sqcup \bar{\mathcal{C}}_m$;
(iv) $\mathcal{B}(i_j, n) = \mathcal{B}_j \sqcup \mathcal{B}_{j+1} \sqcup \cdots \sqcup \mathcal{B}_m$.

Proof. By definition, the unions (iii) and (iv) are disjoint. To prove (iv), note that

$$B(i_j, i_k) = C(i_j, i_k) + (\lambda_{i_k} - \lambda_{i_k+1}) \equiv (\lambda_{i_k} - \lambda_{i_k+1}) \not\equiv 0$$

since i_k is normal, see (2). So $i_k \notin \mathcal{B}(i_j, n)$. If $l \in (i_j, n) \setminus \{i_{j+1}, i_{j+2}, \dots, i_m\}$, we have $l \in (i_k, i_{k+1})$ for some $k \geq j$. Then

$$B(i_j, l) = C(i_j, i_k) + B(i_k, l) \equiv B(i_k, l),$$

which implies (iv). The proof of (iii) is similar.

Recall that i_j is normal by definition. This implies that i_j is normal in $[i_j, i_{j+1})$, i.e. $\mathcal{C}_j$ compensates $\mathcal{B}_j$, which proves (ii). In particular, $|\mathcal{C}_j| \geq |\mathcal{B}_j|$. If $|\mathcal{C}_j| > |\mathcal{B}_j|$, then, by Lemma 1.1, there exists $c \in \mathcal{C}_j$, normal in $[c, i_{j+1})$. Observe that

$$\{l \mid i_{j+1} \leq l < n, \; B(c, l) \equiv 0\} = \mathcal{B}_{j+1} \sqcup \cdots \sqcup \mathcal{B}_m,$$

which is compensated by $\mathcal{C}_{j+1} \sqcup \cdots \sqcup \mathcal{C}_m$. This, together with the fact that c is normal in $[c, i_{j+1})$, implies that c is normal in $[c, n)$. But $C(i_j, c) \equiv 0$ and $i_j < c < i_{j+1}$, which contradicts the choice of $i_1, i_2, \dots, i_m$. $\square$

Now we define a direct summand of $L_n(\lambda)\downarrow_{U(n-1)}$, called the first level, which is of crucial importance for the symmetric groups. Informally, the first level consists of the weight spaces corresponding to the weights obtained from λ by subtracting the last simple root $\alpha(n-1,n)$ and then subtracting an arbitrary linear combination of the other simple roots.

Definition 2.2. The ***first level*** $L^{(1)}$ is

$$L^{(1)} = \oplus L_n(\lambda)_\mu$$

where the sum is over all weights μ such that

$$\lambda - \mu = \sum_{j=1}^{n-2} k_j \alpha(j, j+1) + \alpha(n-1, n)$$

for some $k_1, \dots, k_{n-2} \in \mathbb{Z}_{\geq 0}$.

Since $U(n-1)$ is embedded "by the first $n-2$ simple roots", $L^{(1)}$ is invariant with respect to $U(n-1)$, so it is a submodule, actually a direct summand, of $L_n(\lambda)\downarrow_{U(n-1)}$, see [13] for more details. We always consider $L^{(1)}$ as a $U(n-1)$-module.

To describe a filtration of $L^{(1)}$ we define

$$\begin{aligned} v(j,k) &= T_{i_j,n}\left(\mathcal{C}_j \cup \bar{\mathcal{C}}_{j+1,k} \cup \mathcal{C}_{k+1,m}\right) v_+ \\ &= T_{i_j,n}(\mathcal{C}_{jm} \cup \{i_{j+1}, \dots, i_k\}) v_+ \in L^{(1)}, \ 1 \leq j \leq k \leq m, \end{aligned}$$

and

$$V = \{v(j,k) \mid 1 \leq j \leq k \leq m\}.$$

Remark. It was proved in [14, Theorems 4.2, 4.11] that $v(j,j)$, $j = 1, 2, \dots, m$, are $U(n-1)$-primitive vectors, and the module $U(n-1)v(j,j)$ is irreducible if and only if $j = 1$.

Set

$$Q_\alpha = Q_{\Gamma_\alpha} = Q_\Gamma = U(n-1)\langle V\rangle, \tag{9}$$

the $U(n-1)$-submodule of $L^{(1)}$ generated by V. Below we shall prove that

$$L^{(1)} = \oplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} Q_\alpha$$

and that the non-zero Q_α are indecomposable (by definition, $Q_\alpha = 0$ if and only if there are no normal nodes of residue α). For $r = 1, 2, \dots, 2m-1$ we define sets V_r as follows:

$$V_r = \{v(j,k) \mid 1 \leq j \leq k \leq m, \ 2k - j = r\}.$$

Then

$$V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_{2m-1}.$$

Example. Let $m = 5$. Then

$$V_9 = \{v(1,5)\},\ V_8 = \{v(2,5)\},\ V_7 = \{v(1,4), v(3,5)\},\ V_6 = \{v(2,4), v(4,5)\},$$
$$V_5 = \{v(1,3), v(3,4), v(5,5)\},\ V_4 = \{v(2,3), v(4,4)\},\ V_3 = \{v(1,2), v(3,3)\},$$
$$V_2 = \{v(2,2)\},\ V_1 = \{v(1,1)\}.$$

and

$$\begin{aligned}
v(1,5) &= T_{i_1,n}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_2 \cup \bar{\mathcal{C}}_3 \cup \bar{\mathcal{C}}_4 \cup \bar{\mathcal{C}}_5)v_+,\\
v(2,5) &= T_{i_2,n}(\mathcal{C}_2 \cup \bar{\mathcal{C}}_3 \cup \bar{\mathcal{C}}_4 \cup \bar{\mathcal{C}}_5)v_+,\\
v(1,4) &= T_{i_1,n}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_2 \cup \bar{\mathcal{C}}_3 \cup \bar{\mathcal{C}}_4 \cup \mathcal{C}_5)v_+,\\
v(3,5) &= T_{i_3,n}(\mathcal{C}_3 \cup \bar{\mathcal{C}}_4 \cup \bar{\mathcal{C}}_5)v_+,\\
v(2,4) &= T_{i_2,n}(\mathcal{C}_2 \cup \bar{\mathcal{C}}_3 \cup \bar{\mathcal{C}}_4 \cup \mathcal{C}_5)v_+,\\
v(4,5) &= T_{i_4,n}(\mathcal{C}_4 \cup \bar{\mathcal{C}}_5)v_+,\\
v(1,3) &= T_{i_1,n}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_2 \cup \bar{\mathcal{C}}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(3,4) &= T_{i_3,n}(\mathcal{C}_3 \cup \bar{\mathcal{C}}_4 \cup \mathcal{C}_5)v_+,\\
v(5,5) &= T_{i_5,n}(\mathcal{C}_5)v_+,\\
v(2,3) &= T_{i_2,n}(\mathcal{C}_2 \cup \bar{\mathcal{C}}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(4,4) &= T_{i_4,n}(\mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(1,2) &= T_{i_1,n}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(3,3) &= T_{i_3,n}(\mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(2,2) &= T_{i_2,n}(\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+,\\
v(1,1) &= T_{i_1,n}(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)v_+.
\end{aligned}$$

We now define two important filtrations $(\mathcal{F})$ and (Φ) of Q_Γ. Set

$$(\mathcal{F}) \qquad (0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{2m-1} = Q_\Gamma,$$

where

$$\mathcal{F}_r = U(n-1)\langle V_1, V_2, \dots, V_r\rangle,\ r = 1, 2, \dots, 2m-1.$$

The filtration (Φ) is a refinement of $(\mathcal{F})$ such that every consecutive quotient will turn out to be a module generated by a high weight vector. First we need the following total order on pairs $\{(j,k) \mid 1 \le j \le k \le m\}$: set

$$(10) \qquad (j,k) \prec (j',k') \ \Leftrightarrow\ 2k - j < 2k' - j' \text{ or } 2k - j = 2k' - j' \text{ and } j < j'.$$

Put

$$\Phi_{jk} = U(n)\langle v(j',k') \mid (j',k') \preceq (j,k)\rangle, \quad \Phi'_{jk} = U(n)\langle v(j',k') \mid (j',k') \prec (j,k)\rangle.$$

Then the submodules $\{\Phi_{jk} \mid 1 \le j \le k \le m\}$ taken in our total order form a filtration of Q_Γ with consecutive quotients Φ_{jk}/Φ'_{jk}. We call this filtration (Φ).

We concentrate on $(\mathcal{F})$ first. To prove some of its properties we need the following technical result which will be used to show that certain expressions are equal to zero.

Lemma 2.3. *Let $1 \le l \le m$, $c_u \in \mathcal{C}_l = \{c_1 < c_2 < \cdots < c_d\}$. Then for any $D \subseteq [i_{l+1}, n)$ we have $T_{c_u,n}(\{c_{u+1}, \dots, c_d\} \cup D)v_+ = 0$.*

Proof. Let $\mathcal{B}_l = \{b_1 < b_2 < \cdots < b_d\}$, see Lemma 2.1(i). Then, by Lemma 2.1(ii), $c_j \le b_j$, and $B(c_j, b_j) = B(i_j, b_j) - C(i_j, c_j) \equiv 0$ for $j = u, u+1, \dots, d$. Moreover, $b_j \le i_{l+1} - 2$ (interpret as $b_j \le n-1$ if $l = m$), in view of (8), hence also $c_j \le i_{l+1} - 2$ $(j = u, u+1, \dots, d)$. Now apply Lemma 1.5 with

$i = c_u,\ j = n,\ M = \{c_{u+1}, \dots, c_d\} \cup D,\ k = i_{l+1} - 1$ (interpret as $k = n$ if $l = m$), $M \cap (i, k) = \{c_{u+1}, \dots, c_d\}$, and $\varphi(c_v) = b_v,\ v = u, u+1, \dots, d$.

□

Corollary 2.4. *Let $i_k < t < n$. Then $L_t v(j, k) = 0$.*

Proof. Set

$$M = \mathcal{C}_j \cup \bar{\mathcal{C}}_{j+1,k} \cup \mathcal{C}_{k+1,m}.$$

Then $L_t v(j,k) = L_t T_{i_j,n}(M)\, v_+$. So, in view of Lemma 1.11, we may assume that $t \in \mathcal{C}_{km}$. In this case, again by Lemma 1.11,

$$L_t v(j,k) = -T_{i_j,t}(M \cap (i_j, t)) T_{tn}(M \cap (t, n)) v_+,$$

which is 0 by Lemma 2.3. □

Proposition 2.5. *Let $v(j,k) \in V_r$. Then $v(j,k) + \mathcal{F}_{r-1}$ is a $U(n-1)$-primitive vector in $\mathcal{F}_r / \mathcal{F}_{r-1}$.*

Proof. Note that the weight of $v(j,k)$ is $\lambda - \alpha(i_j, n)$, so we only have to check that $E_{l,l+1} v(j,k) \in \mathcal{F}_{r-1}$ for $l \in [i_j, n-2]$. Recall that $v(j,k) = T_{i_j,n}(M) v_+$, where

$$M = \mathcal{C}_j \cup \{i_{j+1}\} \cup \mathcal{C}_{j+1} \cup \cdots \cup \{i_k\} \cup \mathcal{C}_k \cup \mathcal{C}_{k+1} \cup \cdots \cup \mathcal{C}_m.$$

By Lemma 1.4, we may assume that $l \notin M$ and $l+1 \in M$. Suppose first that $l + 1 \in \{i_{j+1}, \dots, i_k\}$, i.e. $l = i_t - 1$ for $j+1 \le t \le k$. Then

$$M \cap (i_t, l] = \mathcal{C}_t \cup \bar{\mathcal{C}}_{t+1,k} \cup \mathcal{C}_{k+1,m}.$$

So Lemma 1.4 yields

$$E_{l,l+1} v(j,k) = E_{i_t-1,i_t} T_{i_j,n}(M) v_+ = S T_{i_t,n}(M \cap (i_t, n)) v_+ = S v(t,k),$$

for some $S \in U(n-1)$. But $t > j$, hence $2k - t < 2k - j = r$, and so $Sv(t,k)$ belongs to $\mathcal{F}_{r-1}$.

Now, let $l + 1 \in \mathcal{C}_j \cup \mathcal{C}_{j+1} \cup \cdots \cup \mathcal{C}_m$. Then $l + 1 = c_u \in \mathcal{C}_t = \{c_1 < \cdots < c_d\}$, for some $j \le t \le m$.

By Lemma 1.4, we have

$$E_{l,l+1} v(j,k) = E_{c_u-1,c_u} T_{i_j,n}(M) v_+ = S T_{c_u,n}(\{c_{u+1}, \dots, c_d\} \cup D) v_+,$$

where $S \in U(n-1)$ and $D = M \cap [i_{t+1}, n) \subseteq [i_{t+1}, n)$. Now,

$$T_{c_u,n}(\{c_{u+1}, \dots, c_d\} \cup D)v_+ = 0$$

by Lemma 2.3 $\square$

Corollary 2.6. $Q_\Gamma = U(n-1)^-\langle V\rangle$.

The next lemma provides us with a central element $z_{n-1} + x$ which will be used to construct endomorphisms of the first level.

Lemma 2.7. *There exists a constant x such that*

$$(z_{n-1} + x)v(j,k) = v(j,k-1) \quad \textit{if } j < k, \quad \textit{and} \quad (z_{n-1} + x)v(j,j) = 0$$

for all $1 \le j \le k \le m$.

Proof. Recall that $z_{n-1} = \theta_{n-1} - L_2 - L_3 - \dots L_{n-1}$, where $\theta_{n-1} \in U(n-1)^0$. Therefore

$$\theta_{n-1}v(j,k) = (\lambda - \alpha(i_j, n))(\theta_{n-1}).$$

By Lemma 1.12, for any $1 \le j < j' \le m$, we have

$$(\lambda - \alpha(i_j, n))(\theta_{n-1}) - (\lambda - \alpha(i_{j'}, n))(\theta_{n-1}) = C(i_j, i_{j'}) \equiv 0,$$

so we can choose a scalar x (one for all j) such that

$$(\theta_{n-1} + x)v(j,k) = 0, \quad 1 \le j \le k \le m.$$

So

$$\begin{aligned}(z_{n-1} + x)v(j,k) &= (\theta_{n-1} + x - L_2 - L_3 - \dots L_{n-1})v(j,k)\\ &= (-L_2 - L_3 - \dots L_{n-1})v(j,k)\\ &= (C(i_j, i_k) - L_2 - L_3 - \dots L_{n-1})v(j,k)\end{aligned}$$

since $C(i_j, i_k) \equiv 0$. By Lemma 1.11, the last expression equals

$$(C(i_j, i_k) - L_{i_j+1} - L_{i_j+2} - \dots - L_{i_k})v(j,k) - \sum_{t=i_k+1}^{n-1} L_t v(j,k). \tag{11}$$

By Lemma 1.8, we have

$$(C(i_j, i_k) - L_{i_j+1} - L_{i_j+2} - \dots - L_{i_k})v(j,k) = v(j,k-1)$$

(or is absent if $j = k$). So it suffices to apply Corollary 2.4 to show that the second term in (11) equals zero. $\square$

The following result shows that the filtration $(\mathcal{F})$ is as strict as it can be.

Proposition 2.8. *Let $v(j,k) \in V_r$. Then $v(j,k) \notin \mathcal{F}_{r-1}$.*

Proof. If $v(j,k) \in \mathcal{F}_{r-1}$ then, considering weights and using Corollary 2.6, we can write

$$v(j,k) = \sum y_{t,u} v(t,u), \tag{12}$$

where the sum is over all t,u such that $j \leq t \leq u \leq m$, $2u - t < r$, and $y_{t,u} \in U(n)^-_{-\alpha(i_j,i_t)}$.

By Lemma 2.7, there exists a scalar x such that $(z_{n-1}+x)v(j,k) = v(j,k-1)$ if $j < k$ and $(z_{n-1}+x)v(j,j) = 0$. Since z_{n-1} is central in $U(n-1)$, so is $(z_{n-1}+x)$. Multiply both sides of (12) by $(z_{n-1}+x)^{k-j}$. The LHS becomes $v(j,j) = T_{i_j,k}(\mathcal{C}_{jm})v_+ \neq 0$, by Lemma 1.5. On the other hand, since all $y_{t,u} \in U(n-1)$, the RHS equals

$$\sum y_{t,u}(z_{n-1}+x)^{k-j} v(t,u),$$

and we claim that this is zero. Well, in view of Lemma 2.7, we only have to show that $u - t < k - j$. But this is true since $2u - t < r$ implies $u - t < \frac{r-t}{2}$, $2k - j = r$ implies $k - j = \frac{r-j}{2}$, and $t \geq j$. (This argument also shows that if $j = k$ in (12) then there are no t,u such that $j \leq t \leq u \leq m$, $2u - t < r$, and so (12) should be interpreted as $v(j,j) = 0$, which is not true). □

Let us choose x as in Lemma 2.7. Since $z_{n-1} + x$ is a central element of $U(n-1)$ (see (5)), the multiplication by this element defines a $U(n-1)$-endomorphism of the first level $L^{(1)}$. Moreover, in view of Lemma 2.7, the submodule Q_Γ, defined in (9), is invariant with respect to this endomorphism. So we can give the following

Definition 2.9. We define an endomorphism $\varphi_\Gamma \in \mathrm{End}_{U(n-1)}(Q_\Gamma, Q_\Gamma)$ by setting

$$\varphi_\Gamma(v) = (z_{n-1}+x)v, \quad v \in Q_\Gamma.$$

The next result shows that Q_Γ is a cyclic module generated by the vector $v(1,m)$.

Lemma 2.10. $Q_\Gamma = U(n-1)\langle v(1,m)\rangle$.

Proof. Let $2 \leq j \leq m$. By Lemma 2.1(ii) and (8), we have $i_j - 1 \notin \mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,m}$. Since $v(1,m) = T_{i_1,n}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,m})v_+$, Lemma 1.4 yields

$$E_{i_j-1,i_j} v(1,m) = -T_{i_1,i_j-1}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,j-1}) T_{i_j,n}(\mathcal{C}_j \cup \bar{\mathcal{C}}_{j+1,m})v_+. \tag{13}$$

In view of Lemma 1.5, we have $T_{i_1,i_j-1}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,j-1})v_+ \neq 0$, hence there exists an element $u \in U(n-1)^+_{\alpha(i_1,i_j-1)}$ such that

$$uT_{i_1,i_j-1}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,j-1})v_+ = cv_+$$

for a non-zero constant c. Since both u and $T_{i_1,i_j-1}(\mathcal{C}_1 \cup \bar{\mathcal{C}}_{2,j-1})$ commute with $T_{i_j,n}(\mathcal{C}_j \cup \bar{\mathcal{C}}_{j+1,m})$, we conclude from (13) that

$$uE_{i_j-1,i_j} v(1,m) = -cT_{i_j,n}(\mathcal{C}_j \cup \bar{\mathcal{C}}_{j+1,m})v_+ = -cv(j,m).$$

Now, using Lemma 2.7, we get any $v(j,k)$ in $U(n-1)\langle v(1,m)\rangle$, because $(z_{n-1}+x) \in U(n-1)$. □

We are going to use some special $U(n-1)$-weights which are the high weights of certain composition factors of $L^{(1)}$. Set

$$\mu^j = (\lambda - \alpha(i_j, n)) \downarrow \in X(n-1), \quad j = 1, 2, \dots, m. \tag{14}$$

Proposition 2.11.

(i) *Let $1 \leq t < u \leq n$ and $\mu = (\lambda - \alpha(t,u)) \downarrow \in X_+(n-1)$. Then $L_{n-1}(\mu)$ is a composition factor of $L^{(1)}$ if and only if $u = n$ and t is normal;*

(ii) *Let $1 \leq j \leq m$. Then $L_{n-1}(\mu^j)$ is a composition factor of $\mathcal{F}_r/\mathcal{F}_{r-1}$ if and only if $r = 2k - j$ for some $k \in \{j, j+1, \dots, m\}$. In this case the multiplicity $[\mathcal{F}_r/\mathcal{F}_{r-1} : L_{n-1}(\mu^j)]$ is equal to 1.*

Proof. (i) is proved in [16, 9.4].

(ii) Let $j \leq k \leq m$. Put $2k - j = r$. Then $v(j,k) \in \mathcal{F}_r$. By virtue of Propositions 2.8 and 2.5, $v(j,k) + \mathcal{F}_{r-1}$ is a non-zero $U(n-1)^+$-primitive vector of $\mathcal{F}_r/\mathcal{F}_{r-1}$. So $[\mathcal{F}_r/\mathcal{F}_{r-1} : L_{n-1}(\mu^j)] \geq 1$. To prove that the multiplicity is actually equal to 1, and that $L_{n-1}(\mu^j)$ does not appear in the layer $\mathcal{F}_r/\mathcal{F}_{r-1}$ for r not of the form $2k - j$ with $k \geq j$, it suffices to know that

$$[Q_\Gamma : L_{n-1}(\mu^j)] \leq m - (j-1).$$

But this follows from the fact that

$$[L^{(1)} : L_{n-1}(\mu^j)] = m - (j-1),$$

proved in [16, 9.4(ii)]. $\square$

Now we return to the filtration (Φ).

Lemma 2.12. *Let $1 \leq j \leq k \leq m$. Then*

(i) *Φ_{jk}/Φ'_{jk} is a non-zero module generated by a high weight vector $v(j,k) + \Phi'_{jk}$ of weight μ^j;*

(ii) *Φ_{jk}/Φ'_{jk} does not have composition factors of the form $L_{n-1}((\lambda - \alpha(t,u)) \downarrow)$, $1 \leq t < u \leq n$, except $L_{n-1}(\mu^j)$;*

(iii) *$\Phi_{j,k+1}/\Phi'_{j,k+1}$ is a quotient of Φ_{jk}/Φ'_{jk}. Moreover,*

$$\varphi_\Gamma(\Phi_{j,k+1}) \subseteq \Phi_{jk}, \quad \varphi_\Gamma(\Phi'_{j,k+1}) \subseteq \Phi'_{jk},$$

and in this way φ_Γ induces a surjective homomorphism from $\Phi_{j,k+1}/\Phi'_{j,k+1}$ to Φ_{jk}/Φ'_{jk}.

Proof. (i) In view of Proposition 2.5, to prove (i), we only have to show that $v(j,k) \notin \Phi'_{jk}$. Let $2k - j = r$. By Corollary 2.6, we have

$$\Phi'_{jk} = U(n)^- \langle \mathcal{F}_{r-1}, v(j', k') \mid 2k' - j' = r, \ j' < j \rangle$$

However, the weight of $v(j', k')$ is $\lambda - \alpha(i_{j'}, n) < \lambda - \alpha(i_j, n)$, for any $j' < j$. Therefore

$$(\Phi'_{jk})_{\lambda - \alpha(i_j, n)} = (\mathcal{F}_{r-1})_{\lambda - \alpha(i_j, n)}.$$

So $v(j,k) \in \Phi'_{jk}$ implies $v(j,k) \in \mathcal{F}_{r-1}$, which contradicts Proposition 2.8.

(ii) follows from Proposition 2.11 and (i).

(iii) In view of Lemma 2.7, $\varphi_\Gamma(\Phi_{j,k+1}) \subseteq \Phi_{jk}$ and $\varphi_\Gamma(\Phi'_{j,k+1}) \subseteq \Phi'_{jk}$. So φ_Γ induces a homomorphism from $\Phi_{j,k+1}/\Phi'_{j,k+1}$ to Φ_{jk}/Φ'_{jk}. It is surjective by (i), since $\varphi_\Gamma(v(j,k+1)) = v(j,k)$ in view of Lemma 2.7. $\square$

Next we want to show that Q_Γ's are actually direct summands of $L^{(1)}$. We now return to the more detailed notation with α's in the indices: $\Gamma_\alpha, Q_\alpha, i_j^{(\alpha)}$, etc. Recall that, by definition, $Q_\alpha \neq 0$ if and only if there is a normal i with $\mathrm{res}(i, \lambda_i) = \alpha$.

Theorem 2.13.

(i) $L^{(1)} = \oplus_{\alpha\in\mathbb{Z}/p\mathbb{Z}} Q_\alpha$;
(ii) *If $Q_\alpha \neq 0$ then Q_α has a simple socle and a simple head both isomorphic to $L_{n-1}((\lambda - \alpha(i_1^\alpha, n))\downarrow)$;*
(iii) *Q_α is indecomposable (for any α).*
(iv) *Q_α and Q_β are in distinct blocks (for any pair $\alpha \neq \beta$);*
(v) *Q_α is (contravariantly) selfdual (for any α).*

Proof. Put $M = \sum_\alpha Q_\alpha$. We first show that $M = L_1$. By (1), it suffices to show that every $F_{jn}v_+$, $j = 1, 2, \ldots, n-1$, is contained in M. We apply downward induction on j. Let $j = n-1$. If $\lambda_{n-1} - \lambda_n = \lambda_{n-1} \equiv 0$ then $F_{n-1,n}v_+ = 0$. Otherwise $n-1$ is normal, so $n-1$ is equal to some $i^\alpha_{m_\alpha}$, i.e. belongs to the corresponding Γ_α. Now $F_{n-1,n}v_+ = T_{n-1,n}(\emptyset)v_+ = v(m_\alpha, m_\alpha) \in Q_\alpha \subseteq M$. Let $j < n-1$. If j is not normal then $F_{j,n}v_+$ can be expressed through non-trivial subdivisions, see Lemma 1.7, and the inductive hypothesis applies. So we may assume that j is normal. Then j belongs to some Γ_α of the form (7). Let $j = i^\alpha_t$. By Lemmas 2.1(iii) and 1.2, we have for the corresponding vector $v(t, m_\alpha)$:

$$v(t, m_\alpha) = aF_{jn}v_+ + [\text{other terms}]$$

for some non-zero constant a. Applying inductive hypothesis to [other terms] and using the fact that $v(t, m_\alpha) \in Q_\alpha$, we conclude that $F_{jn}v_+ \in Q_\alpha \subseteq M$.

Now, let $\alpha(1), \ldots, \alpha(q)$ be all the residues α such that $Q_\alpha \neq 0$ (equivalently, $\Gamma_\alpha \neq \emptyset$). It is proved in [14, 4.11] that the socle of $L^{(1)}$ is

$$\oplus_{j=1}^q L_{n-1}((\lambda - \alpha(i_1^{\alpha(j)}, n))\downarrow).$$

Note that the Young diagrams of the weights $(\lambda - \alpha(i_1^{\alpha(j)}, n))\downarrow$ all have different residue contents, just because all $\alpha(j)$ are different. By the linkage principle for $GL(n)$ (see the explanation in the proof of [5, 4.3]), all composition factors of the socle belong to different blocks. This immediately implies that all $Q_{\alpha(j)}$ are indecomposable, belong to different blocks, and have the simple socles as claimed in (ii).

Moreover, it is proved in [11] (see also [12, 2.9]) that $L^{(1)}$ is (contravariantly) selfdual. (We refer the reader to [10, p.205] for the definition of the contravariant dual ${}^\tau V$ of a module V.) Since τ preserves the blocks, all ${}^\tau Q_\alpha$ are also in different blocks, and we may conclude that ${}^\tau Q_\alpha \cong Q_\alpha$. This completes the proof of the theorem. $\square$

Theorem 2.14. *Let*

$$\Gamma = \Gamma_\alpha = \{i_1 < i_2 < \cdots < i_m\}.$$

Then

$$\{\mathrm{id}_{Q_\Gamma}, \varphi_\Gamma, \dots, \varphi_\Gamma^{m-1}\}$$

is a basis of $\mathrm{End}_{U(n-1)}(Q_\Gamma, Q_\Gamma)$.

Proof. Let us write φ and Q for φ_Γ and Q_Γ. We first prove that the homomorphisms $\varphi^0 = \mathrm{id}_Q, \varphi, \dots, \varphi^{m-1}$ are linearly independent. Assume a linear combination

$$c_0\varphi^0 + c_1\varphi + \cdots + c_{m-1}\varphi^{m-1}$$

is zero. If not all c_t are equal to zero, choose the smallest number s with $c_s \neq 0$. By Lemma 2.7, we have

$$\varphi^s(v(1,m)) = v(1, m-s),$$

and

$$\varphi^t(v(1,m)) = v(1, m-t) \in \mathcal{F}_{2(m-t)-1}.$$

In view of Proposition 2.8, we have $v(1, m-s) \notin \mathcal{F}_{2(m-t)-1}$ for any $t > s$. So

$$(c_0\varphi^0 + c_1\varphi + \cdots + c_{m-1}\varphi^{m-1})(v(1,m)) \neq 0,$$

giving a contradiction.

Now, we show that $\{\varphi^0, \varphi, \dots, \varphi^{m-1}\}$ span the space $\mathrm{End}_{U(n-1)}(Q)$. Pick an element $f \in \mathrm{End}_{U(n-1)}(Q)$. There exists a pair (j,k) with $1 \leq j \leq k \leq m$ such that $f(v(1,m)) \in \Phi_{jk} \setminus \Phi'_{jk}$. We prove the result by induction on the (j,k), taken in the total order (10), using the case $f(v(1,m)) = 0$ as the induction base. Indeed, if $f(v(1,m)) = 0$ then $f = 0$ by Lemma 2.10.

By Theorem 2.13, Q has a simple head isomorphic to $L_{n-1}(\mu)$ where $\mu = (\lambda - \alpha(i_1, n)) \downarrow$. Let J be the submodule of Q such that $Q/J \cong L_{n-1}(\mu)$. Then $v(1,m)+J$ is a (non-zero) high weight vector of Q/J.

We claim that $L_{n-1}(\mu)$ is a composition factor of Φ_{jk}/Φ'_{jk}. Indeed, $f(Q)$, being a non-zero quotient of Q, has the same simple head, and $f(v(1,m)) + f(J)$ is a non-zero high weight vector of $f(Q)/f(J)$. Since $f(J)$ is the unique maximal proper submodule of $f(Q)$, we have $f(J) \supseteq f(Q) \cap \Phi'_{jk}$. We conclude that the module

$$(f(Q) + \Phi'_{jk})/\Phi'_{jk} \cong f(Q)/(f(Q) \cap \Phi'_{jk})$$

contains a composition factor $L_{n-1}(\mu)$. But this module is a submodule of Φ_{jk}/Φ'_{jk} because $f(v(1,m)) \in \Phi_{jk}$ and $v(1,m)$ generates Q in view of Lemma 2.10.

Now, by Lemma 2.12, we have $j = 1$ and

$$f(v(1,m)) \cong cv(1,k) \pmod{\Phi'_{1k}}$$

for some non-zero constant c. However $\varphi^{m-k}(v(1,m)) = v(1,k)$, so the inductive hypothesis applies to $f - c\varphi^{m-k}$. $\square$

3. From $GL(n)$ to Σ_n.

The information about Schur functors used in this section can be found in [8], [11] and [13]. Let $M_F(n,n)$ be the category of the (left) finite dimensional polynomial representations of $GL(n)$ in which all coefficients are homogeneous polynomials of degree n in the matrix coefficients of $GL(n)$. We also denote by $\mod {}_F\Sigma_n$ the category of the (left) finite dimensional $F\Sigma_n$-modules. If $V \in M_F(n,n)$ and $\omega = (1,1,\dots,1) \in \Lambda(n)$ then the weight space V_ω is invariant with respect to the subgroup of monomial matrices in $GL(n)$. This subgroup is isomorphic to Σ_n, and so $V \to V_\omega$ defines an exact functor

$$f_n : M_F(n,n) \to \mod {}_F\Sigma_n.$$

Under this functor, a homomorphism $\varphi : V \to W$ goes to $f_n(\varphi) : V_\omega \to W_\omega$ which is just the restriction of φ to V_ω.

A dominant weight $\lambda \in X_+(n)$ is called ***p-restricted*** if $\lambda_i - \lambda_{i+1} < p$ for $1 \le i < n$. It is known that

$$f_n(L_n(\lambda)) = \begin{cases} D^{\lambda'} \otimes \mathrm{sgn}_n & \text{if } \lambda \text{ is } p\text{-restricted}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$f_n(\Delta_n(\lambda)) = S^{\lambda'} \otimes \mathrm{sgn}_n,$$

where λ' denotes the partition conjugate to λ, and sgn_n is the alternating representation of Σ_n.

It was first noticed in [11] (see also [12, 4.7(i),4.8]) that the first level $L^{(1)}$ is the 'right object' to look at if we are interested in restriction from Σ_n to Σ_{n-1}:

$$L^{(1)} \in M_F(n-1,n-1), \text{ and } f_{n-1}(L^{(1)}) = (D^{\lambda'} \otimes \mathrm{sgn}_n) \downarrow_{\Sigma_{n-1}} .$$

In [11], a certain subcategory $M_F(n,n)^c$ of $M_F(n,n)$ was shown to be equivalent to $\mod {}_F\Sigma_n$ under the Scur functor and certain inverse functor

$$h_n : \ \mod {}_F\Sigma_n \to M_F(n,n)^c.$$

This subcategory can be defined as a full subcategory of $M_F(n,n)$ having p-restricted socles and heads. It is shown in [12, Theorem B] that $L^{(1)}$ is such a module. This explains how to deduce Theorems A(i), B, E in the Introduction from Theorem 2.14 and Lemma 2.12.

4. On a Conjecture of Benson

In this section we show how to deduce Theorems A(ii), C and Corollary D from Theorem A(i).

Lemma 4.1. *Let V and W be $F\Sigma_n$-modules. Then*

$$\begin{aligned} &\dim \mathrm{Hom}_{\Sigma_{n+1}}(V \uparrow^{\Sigma_{n+1}}, W \uparrow^{\Sigma_{n+1}}) \\ &= \dim \mathrm{Hom}_{\Sigma_n}(V,W) + \dim \mathrm{Hom}_{\Sigma_{n-1}}(V \downarrow_{\Sigma_{n-1}}, W \downarrow_{\Sigma_{n-1}}). \end{aligned}$$

Proof. This is the Intertwining Number Theorem from [7, 44.5] with $G = \Sigma_n$, $H_1 = H_2 = \Sigma_{n-1}$. $\square$

Corollary 4.2. *Let D^λ be an irreducible $F\Sigma_n$-module. Then*

$$\dim \mathrm{End}_{\Sigma_{n+1}}(D^\lambda \uparrow^{\Sigma_{n+1}}) = 1 + \dim \mathrm{End}_{\Sigma_{n-1}}(D^\lambda \downarrow_{\Sigma_{n-1}}).$$

This corollary together with Theorem A(i) yields Theorem A(ii). To deduce Theorem C it now suffices to prove the following

Lemma 4.3. *Let D^λ be an irreducible $F\Sigma_n$-module. Suppose λ has r good addable nodes, say, $B_1, B_2, \dots, B_r$. Then $D^\lambda \uparrow^{\Sigma_{n+1}}$ is semisimple if and only if*

$$\dim \mathrm{End}_{\Sigma_{n+1}}(D^\lambda \uparrow^{\Sigma_{n+1}}) = r.$$

Moreover, in this case we have

$$D^\lambda \uparrow^{\Sigma_{n+1}} \cong D^{\lambda^{B_1}} \oplus D^{\lambda^{B_2}} \oplus \cdots \oplus D^{\lambda^{B_r}}.$$

Proof. By [15, 3.2], we have

$$D^\lambda \uparrow^{\Sigma_{n+1}} \cong J_1 \oplus J_2 \oplus \cdots \oplus J_r$$

where every J_i is indecomposable and has a simple socle and a simple head both isomorphic to $D^{\lambda^{B_i}}$. So $\dim \mathrm{End}_{\Sigma_{n+1}}(J_i) = 1$ if J_i is irreducible, and $\dim \mathrm{End}_{\Sigma_{n+1}}(J_i) > 1$ otherwise. The lemma follows. $\square$

Finally we prove Corollary D.

Corollary 4.4. *Let $p = 2$, $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_t > 0)$ be a 2-regular partition, and D^λ be the corresponding irreducible. Then $D^\lambda \uparrow^{\Sigma_{n+1}}$ is semisimple if and only if all λ_i are even. In this case,*

$$D^\lambda \uparrow^{\Sigma_{n+1}} \cong D^{(\lambda_1, \dots, \lambda_{t-1}, \lambda_t + 1)} \oplus D^{(\lambda_1, \dots, \lambda_t, 1)}.$$

Proof. If all λ_i are even then the top removable node $(1, \lambda_1)$ is the only normal node for λ, and the two bottom addable nodes, $(t+1, 1)$ and $(t, \lambda_t + 1)$, are the only good addable nodes for λ. In view of Theorem C, this proves the result in one direction.

Conversely, let $D^\lambda \uparrow^{\Sigma_{n+1}}$ be semisimple. In characteristic 2 a partition has at most two good addable nodes, see [15, 1.4]. In our case there should be exactly two, since otherwise, in view of Theorem C, $D^\lambda \uparrow^{\Sigma_{n+1}}$ is irreducible, which implies $\dim \mathrm{End}_{\Sigma_{n+1}}(D^\lambda \uparrow^{\Sigma_{n+1}}) = 1$, contradicting Corollary 4.2.

So we have to show that if λ has two good addables and one normal removable then all λ_i are even. But a 2-regular partition has only one removable if and only if $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_t \pmod 2$. If all λ_i are odd then the only good addable node is the next to the bottom one. This proves the corollary. $\square$

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